MATH 524: Lecture 7 (09/09/2025)

Today: * chains
* boundary homomorphism, fundamental lemma of homology

Recall orientation, $\sigma = [v_0 \dots v_p] \to$ oriented simplex, opposite orientation...

Starting with oriented simplices, we define collections of them as functions, and then consider adding these functions — to define groups.

Def Let K be a simplicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to $\mathbb{Z}$ such that

(1) $c(\sigma) = -c(\sigma')$ if σ, σ' are opposite orientations of the same simplex; and

(2) $c(\sigma) = 0$ for all but finitely many p-simplices σ.

Thus, even on infinite simplicial complexes, each p-chain has nonzero values on only finitely many p-simplices.

We can add two p-chains by adding their values. The resulting group is the group of oriented p-chains of K, $C_p(K)$. If $p<0$ or $p>\dim(K)$, $C_p(K)$ is trivial.

One can indeed check that $C_p(K)$ is a group - identity (0), inverse ($c(\sigma')$), and associativity all hold. In fact $C_p(K)$ are abelian groups, as adding the functions is commutative.

Are there really only two orientations of higher dimensional simplices?

YES! Consider a 3-simplex $\sigma = [v_0 v_1 v_2 v_3]$ and $-\sigma = [v_1 v_0 v_2 v_3]$, its reverse orientation. What about $[v_2 v_0 v_3 v_1]$, for instance?

$$[v_2 v_0 v_3 v_1] \rightarrow [v_0 v_2 v_1 v_3] \rightarrow [v_0 v_1 v_2 v_3]$$ 3 swaps, i.e., odd.

Hence $[v_2 v_0 v_3 v_1]$ should be the opposite orientation to $[v_0 v_1 v_2 v_3]$. But then it should be the same orientation as $[v_1 v_0 v_2 v_3]$.

check: $[v_2 v_0 v_3 v_1] \rightarrow [v_1 v_0 v_3 v_2] \rightarrow [v_1 v_0 v_2 v_3]$ 2 swaps, i.e., even!

For oriented simplex σ, the **elementary chain** c corresponding to σ is the function defined as follows:

$c(\sigma) = 1$,

$c(\sigma') = -1$, where σ' is the opposite orientation of σ,

$c(\tau) = 0$, $\forall\, \tau \neq \sigma$.

The correspondence to unit vectors in a Euclidean space is indeed direct here. The elementary chains have value $+1$ for exactly one p-simplex. Later on, we will see that these elementary chains correspond to unit vectors representing each p-simplex — at least in the case when K is finite.

Notation: σ denotes the simplex, oriented simplex, or the elementary chain corresponding to the simplex. Then we can write $\sigma' = -\sigma$ (where σ' is the simplex with orientation opposite to that of σ).

Lemma 5.1 [M] $C_p(K)$ is free abelian, and a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains.

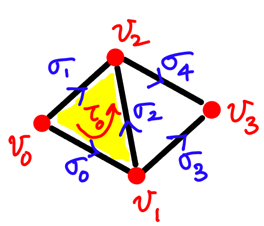
Notice that $C_0(K)$ has a "natural" basis, since each 0-simplex has only one orientation. But we do need to choose an orientation for each p-simplex to get a basis when $p > 0$. And there exist many bases when $p > 0$.

Corollary [M] Any function f from oriented p-simplices of K to abelian group G extends naturally to a homomorphism from $C_p(K)$ to G provided $f(-\sigma) = -f(\sigma)$ for all oriented p-simplices σ in K.

reverse orientation of σ.

Let's consider a small example.

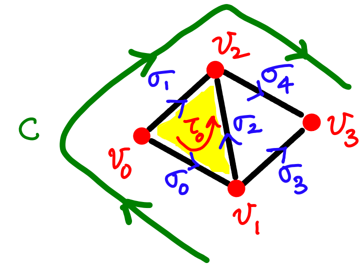
K:



Let K be as shown here, with 4 vertices (v_0, v_1, v_2, v_3), 5 edges $(\sigma_0 - \sigma_4)$, and 1 triangle (τ_0). We orient the edges lexicographically, i.e., $[v_i v_j]$ with $i<j$. The triangle τ_0 is oriented as $[v_0 v_1 v_2]$, or CCW as shown here.

A 1-chain c can be specified as follows:

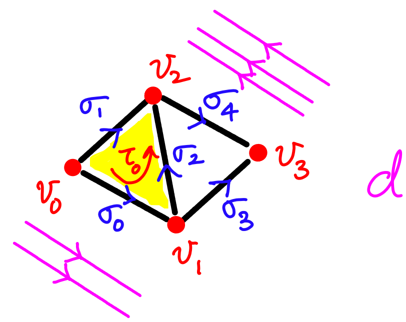
$c(\sigma_0) = -1$
$c(\sigma_1) = 1$ $\quad c(\sigma_2) = 0,$
$c(\sigma_4) = 1$ $\quad c(\sigma_3) = 0$



Notice that a value of -1 on σ_0 could be interpreted as "traversing" σ_0 once in its reverse orientation, i.e., going from v_1 to v_0 once. Following the same logic, we see that c here represents the piecewise linear "curve" going $v_1 \to v_0 \to v_2 \to v_3$ (once).

Here is another 1-chain d:

$d(\sigma_0) = 2$, $d(\sigma_4) = -3$
$d(\sigma_j) = 0$, $j = 1, 2, 3$.



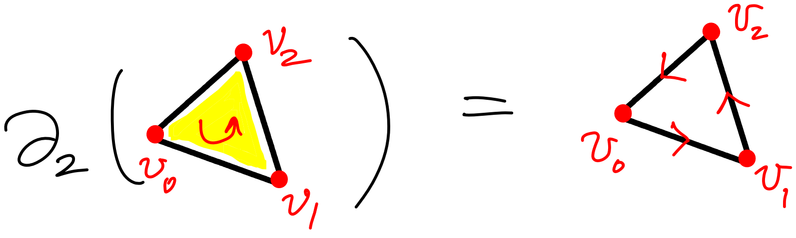
In particular, notice that the chains need not represent single connected pieces all the time.

A 2-chain can be $g(\tau_0) = 2$, which represents two copies of the single triangle in K.

Now that we have defined the chain groups $C_p(K)$ for each p, we now talk about how to connect/relate the $C_p(K)$ for various p. In particular, how are $C_p(K)$ and $C_{p-1}(K)$ related?

We define a homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ → Called the "p-boundary"
called the **boundary operator** (or boundary homomorphism).

Intuitively, the boundary of a triangle is made of its three edges. But now we take the orientation also into account.



Def We define the homomorphism

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K) \text{ called the}$$

boundary operator as follows. If $\sigma = [v_0, \ldots, v_p]$, $p > 0$, then

$$\partial_p \sigma = \partial_p [v_0, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p] \qquad (1)$$

where $\hat{v}_i$ means vertex v_i is deleted from $[v_0, \ldots, v_p]$.

As $C_p(K)$ is trivial for $p < 0$, ∂_p is the trivial homomorphism for $p \le 0$.

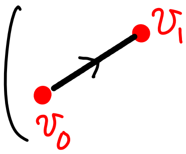
Since ∂_p is a homomorphism, we naturally extend the definition of boundary from p-simplices to p-chains. If $c = \sum n_i \sigma_i$ is a p-chain, then $\partial_p c = \partial_p\left(\sum n_i \sigma_i\right) = \sum n_i (\partial_p \sigma_i)$.

Examples

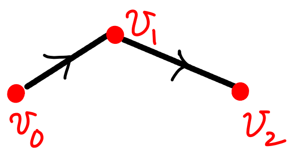
1-simplex

$\partial_1 [v_0 v_1] = v_1 - v_0$

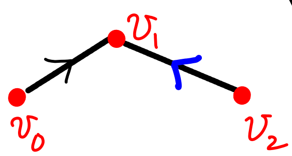
Notice that $\partial_1 [v_1 v_0] = v_0 - v_1$;

$\partial_1 \Bigg($  $\Bigg) = v_1 - v_0$

head − tail, if you think of the oriented edge as an "arrow".

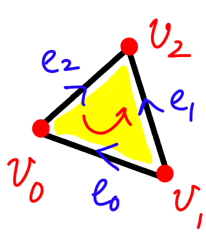
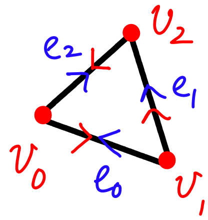
$\partial_1 \Bigg($  $\Bigg) = v_1 - v_0 + v_2 - v_1 = v_2 - v_0$

Notice that the computations are sensitive to the choice of orientations.

$\partial_1 \Bigg($  $\Bigg) = 2v_1 - v_0 - v_2$. $\quad (v_1 - v_0 + v_1 - v_2)$

2-simplex

$$\partial_2 [v_0 v_1 v_2] = (-1)^0 [v_1 v_2] + (-1)^1 [v_0 v_2] + (-1)^2 [v_0 v_1] = [v_1 v_2] - [v_0 v_2] + [v_0 v_1].$$

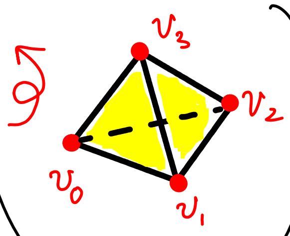
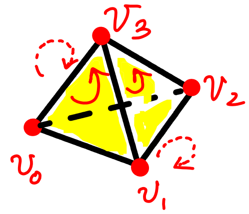
$\partial_2 \Bigg($  $\Bigg) =$ 

The 1-boundary is $-e_0 + e_1 - e_2$

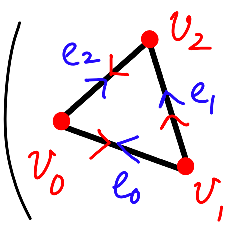
Notice that the orientation induced from the 2-simplex onto its faces (1-simplices) by the boundary operation could be distinct from the individual orientations of the 1-simplices themselves.

3-simplex

$$\partial_3 [v_0v_1v_2v_3] = [v_1v_2v_3] - [v_0v_2v_3] + [v_0v_1v_3] - [v_0v_1v_2]$$

$\partial_3\Big($  $\Big) =$ 

We observe that $\partial_1(\partial_2[v_0v_1v_2]) = 0$. (both algebraically and geometrically)

$\partial_1\Big($  $\Big) = \partial_1(-e_0+e_1-e_2) = -(v_0-v_1)+(v_2-v_1)-(v_2-v_0) = 0.$

A similar observation can be made for the tetrahedron:

$$\partial_2(\partial_3 [v_0v_1v_2v_3]) = \partial_2\big([\underbrace{v_1v_2}_{+[v_1v_2]}v_3] - [v_0v_2v_3] + [v_0v_1v_3] - [v_0\underbrace{v_1v_2}_{-[v_1v_2]}]\big) = 0$$

every edge cancels in pairs.

Indeed, this result holds in general — $\partial_p\partial_{p+1}\sigma = 0$. And we can prove it using the definition of ∂_p.

Before that, let's make sure ∂_p is well-defined. In particular, we need to check that $\partial_p(-\sigma) = -\partial_p(\sigma)$.

We check what happens in Sum (1) when we swap v_j & v_{j+1}.

Consider $\sum_{i=0}^{p}(-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_p]$ and $\sum_{i=0}^{p}(-1)^i(-[v_0,\ldots,\hat{v}_i,\ldots v_p])$. If $i\neq j,j+1$, the corresponding terms do differ by a sign. When $i=j$, compare terms in

$$\partial_p[v_0,\ldots,v_{j-1},v_j,v_{j+1},v_{j+2},\ldots,v_p] \text{ ——— (1a)}$$

and swapped

$$\partial_p[v_0,\ldots,v_{j-1},v_{j+1},v_j,v_{j+2},\ldots,v_p]. \text{ ——— (1b)}$$

before we leave out one vertex at a time...

We have $(-1)^j[v_0,\ldots v_{j-1},\hat{v}_j,v_{j+1},v_{j+2},\ldots,v_p]$ in (1a), and

$(-1)^{j+1}[v_0,\ldots,v_{j-1},v_{j+1},\hat{v}_j,v_{j+2},\ldots,v_p]$ in (1b)

These two terms do differ by a sign: $(-1)^j$ and $(-1)^{j+1}$. Argument for $i=j+1$ is similar.

We now prove the general result on taking the boundary of a boundary. Indeed, we will use this result to define homology groups as subgroups of $C_p(K)$. Hence this result is called the fundamental lemma of homology.

Lemma 5.3 [M] $\quad \partial_{p-1}\circ\partial_p = 0. \longrightarrow$ Fundamental lemma of homology

Proof

$$\begin{aligned}
&\partial_{p-1}\partial_p[v_0,\ldots,v_p]\\
&=\sum_{i=0}^{p}(-1)^i\partial_{p-1}[v_0,\ldots,\hat{v}_i,\ldots v_p]\\
&=\sum_{j<i}(-1)^i(-1)^j[\ldots\hat{v}_j,\ldots,\hat{v}_i,\ldots]+\sum_{j>i}(-1)^i(-1)^{j-1}[\ldots\hat{v}_i,\ldots\hat{v}_j\ldots]\\
&=0, \text{ as the terms cancel in pairs!}
\end{aligned}$$

$\square$