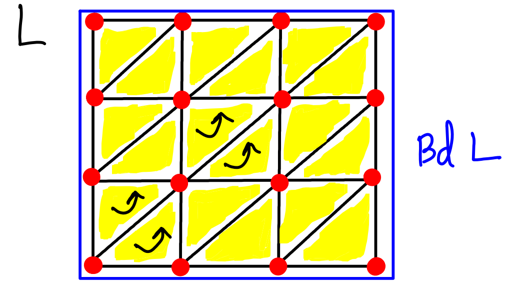
MATH 524 : Lecture 9 (09/16/2025)

Today: * Homology groups of torus, Klein bottle

Lemma 6.1 [M]



Let L be the simplicial complex such that $|L|$ is a rectangle. Let $\mathrm{Bd}\,L$ be the subcomplex of L representing the edges making up the boundary of the rectangle.

Orient each triangle counter clockwise (CCW), and the edges arbitrarily. Then the following results hold.

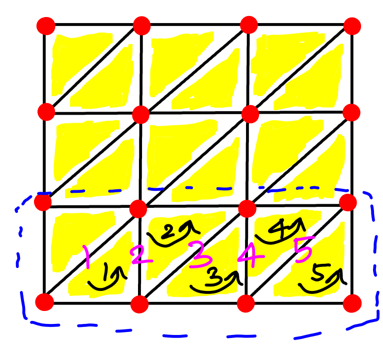
(1) Every 1-cycle of L is homologous to a 1-cycle carried by $\mathrm{Bd}\,L$.

(2) If $\bar{d}$ is a 2-chain of L, and if $\partial \bar{d}$ is carried by $\mathrm{Bd}\,L$, then $\bar{d}$ is a multiple of the chain $\sum \sigma_i$, where σ_i are all the triangles.

Proof of Lemma 6.1 [M]

(2) follows because if σ_i and σ_j in $\bar{d}$ have an edge in the middle in common, then $\partial \bar{d}$ has coefficient 0 on that edge. Hence, $\bar{d}$ has the same value on σ_i & σ_j. We can extend this argument to all σ_i's, as $\partial \bar{d}$ is carried by $\mathrm{Bd}\,L$, giving that $\bar{d}$ has the same coefficient on all of them.

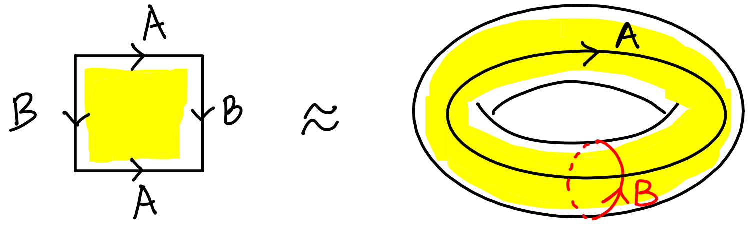
(1) We use the idea of pushing the chain off of edges in the middle (as we did in Example 3).



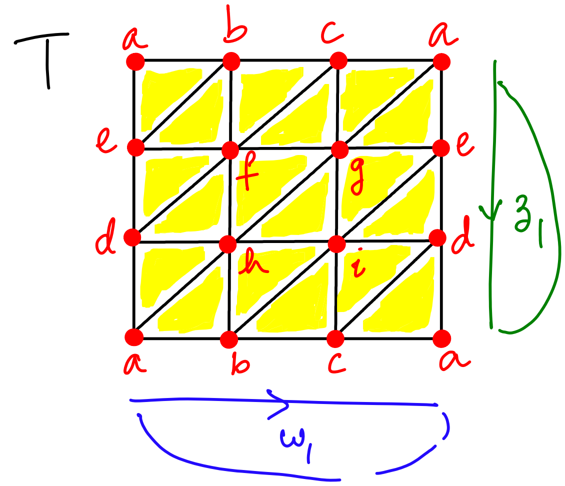
We can use the triangles in the order shown here—from 1a to 5a—to push the input chain off of edges marked 1 to 5 (in that order). After these steps, the chain will be pushed on to the edges shown with the dashed outline.

We then repeat the process on the second "horizontal strip" of triangles, and then the top strip. Ultimately, the chain is carried by $\mathrm{Bd}\,L$. □

Theorem 6.2 [M] $(\mathbb{T}^2)$
(homology of torus)



Let T be the simplicial complex representing L, the rectangle, along with the vertex labels. $|T|$ is the torus.



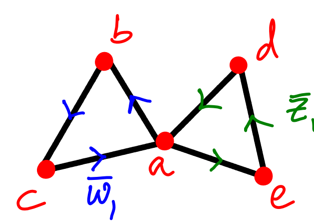
Then $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_2(T) \simeq \mathbb{Z}$.

Orient each 2-simplex CCW; let $\bar{\gamma}$ denote their sum (2-chain). Let $\bar{w}_1 = [a,b] + [b,c] + [c,a]$, and $\bar{z}_1 = [a,e] + [e,d] + [d,a]$.

Then $\bar{\gamma}$ generates $H_2(T)$ and $\{\bar{w}_1, \bar{z}_1\}$ is a basis for $H_1(T)$.

Proof

Let $g: |L| \to |T|$ be the pasting map (labeling), and let $A = g(|\mathrm{Bd}\, L|)$. Then A is homeomorphic to the union of two circles with one point in common (also called a wedge of two circles).



We apply the same "pushing off" arguments as in the proof of the last Lemma. We get the following results.

(1) Every 1-cycle of T is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of T and $\partial\bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

Notice the correspondence of statements (1) and (2) above to those in Lemma 6.1 [M].

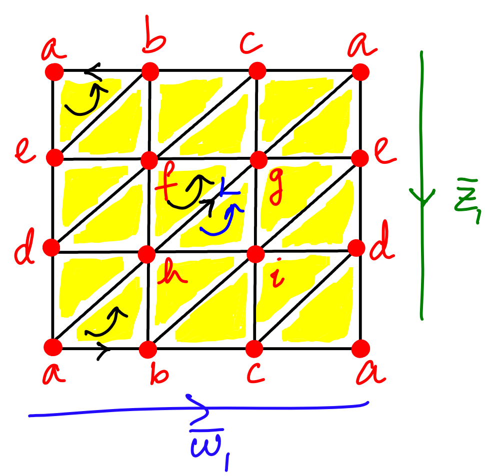
But we get two more results here.

(3) If $\bar{c}$ is a 1-cycle carried by A, then $\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$; and

(4) $\partial_2 \bar{r} = 0$.

To see these results, check the orientations induced on each edge by the two triangles it is a face of.

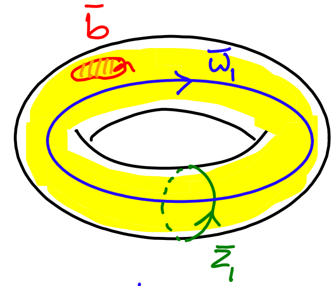
e.g., $[a,b]$ has $+1$ from $\partial_2[abh]$ and -1 from $\partial_2[aeb]$.



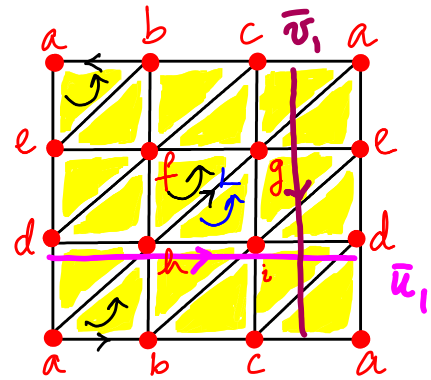
A 1-cycle in A bounds only if it is trivial, as every 1-cycle is $\bar{c} = m\bar{w}_1 + n\bar{z}_1$, and if $\bar{c} = \partial_2\bar{d}$ then by (2) $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. Hence by (4), since $\partial_2\bar{d} = 0$, we must have $m=n=p=0$ here.

Hence we can conclude that $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.

Imagine elastic bands wrapping around the torus along $\bar{w}_1$ and $\bar{z}_1$ as shown here. These bands cannot be shrunk to a point, while another cycle/band represented by $\bar{b}$ (as shown) can be shrunk to a point. Indeed $\bar{b}$ bounds the patch of surface it encloses.

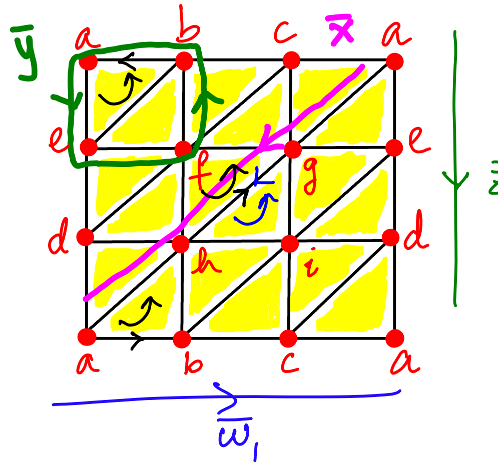


Also, there are many other choices for a basis for $H_1(T)$. For instance, we could use $\{\bar{u}_1, \bar{v}_1\}$, where $\bar{u}_1 = [d,h]+[h,i]+[i,d]$ and $\bar{v}_1 = [c,g]+[g,i]+[i,c]$.

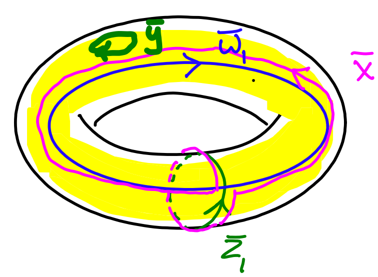


What about the cycle $\bar{x} = [a,g]+[g,h]+[h,a]$?

We can see that $\bar{w}_1 - \bar{z}_1 + \bar{x}$ is the boundary of the 2-chain consisting of all triangles below $\bar{x}$ in the diagram. In other words, $\bar{x}$ is homologous to $\bar{w}_1 + (-1)\bar{z}_1$, or equivalently, we can write $\bar{x}$ as a combination of $\bar{w}_1$ and $\bar{z}_1$.

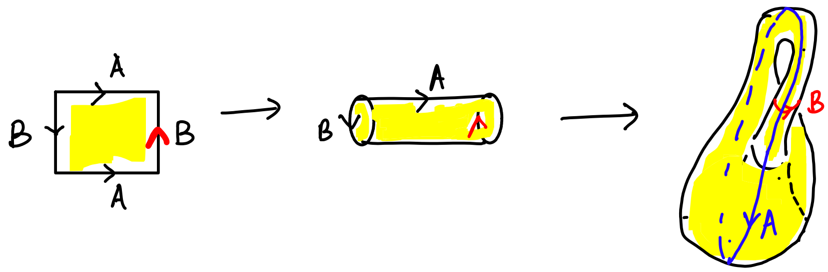


And, $\bar{y}$ is a boundary — it is the boundary of the 2-chain consisting of triangles aeb and bef. So $\bar{y} \in B_1(T)$, and hence $\notin H_1(T)$.

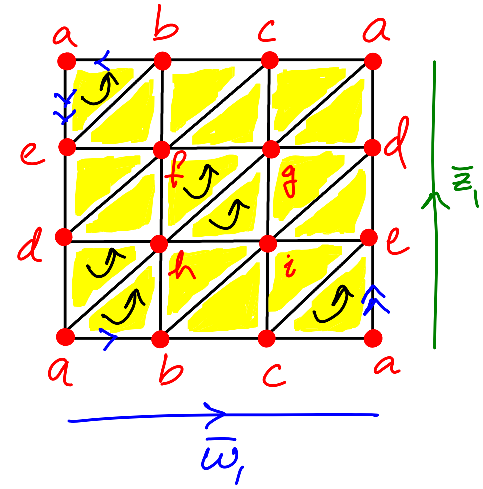


$H_2(T)$: By (1) and (2), if $\bar{d}$ is a 2-cycle of T then $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. But (4) says $\partial_2 \bar{r} = 0$, and hence every such 2-chain is a 2-cycle. There are no tetrahedra in T, and hence no 2-boundaries. So, $H_2(T) = Z_2(T) \simeq \mathbb{Z}$, with $\bar{r}$ being a generator. $\square$

Theorem 6.3 [M] (Klein bottle)



K



Let K be the complex shown, and L the rectangle with the labels. $|K| \approx$ Klein bottle. Then

$$H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad H_2(K) = 0.$$

Let
$$\bar{w}_1 = [a,b] + [b,c] + [c,a] \quad \text{and}$$
$$\bar{z}_1 = [a,e] + [e,d] + [d,a]$$

Then the torsion subgroup of $H_1(K)$ is represented by $\bar{z}_1$, and the free part is generated by $\bar{w}_1$.

Proof

We follow the same technique as in the case of the torus. Indeed, we can push an input chain off of all the edges in the middle, as before. But notice that the edges in A (boundary of the square) do not all behave identically. For instance, $[a,b]$ does get opposite orientations from $\partial_2[abh]$ and $\partial_2[aeb]$. But $[a,e]$ gets $+1$ from both $\partial_2[aec]$ and $\partial_2[aeb]$. As such, $\partial_2 \bar{r} \neq 0$ here!

Similar to the previous case, we get the following results.

(1) Every 1-cycle of K is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of K and $\partial\bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

We also get (3):

(3) If $\bar{c}$ is a 1-cycle carried by A, then $\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$.

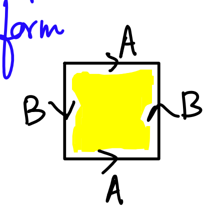
But instead of (4), we get

(4') $\partial_2\bar{r} = 2\bar{z}_1$.

Like last time, we get that $\bar{c}$, a 1-cycle of K, is homologous to $m\bar{w}_1 + n\bar{z}_1$. If $\bar{c} = \partial_2\bar{d}$, then $\bar{c} = \partial_2\bar{d} = 2p\bar{z}_1$, $p \in \mathbb{Z}$. Hence, $\bar{c}$ is a boundary iff $m = 0$, n is even.

Hence we get $H_1(K) \simeq \underbrace{\mathbb{Z}}_{\text{free}} \oplus \underbrace{\mathbb{Z}/2}_{\text{torsion}}$, with $\bar{w}_1$ generating the free part and $\bar{z}_1$ generating the torsion part.

Intuitively, one can see from the "pasting" picture itself that the boundary of the square space is $2B$. In the case of the torus, both A and B do not form boundaries, but here, $2B$ is the boundary.



$\underline{H_2(K)}$: Since $\partial_2\bar{r} = 2\bar{z}_1 \neq 0$, $Z_2(K) = 0$, and hence $H_2(K) = 0$.

Intuitively, the Klein bottle does not enclose a 3D space like the torus. Hence, its 2nd homology group is trivial.