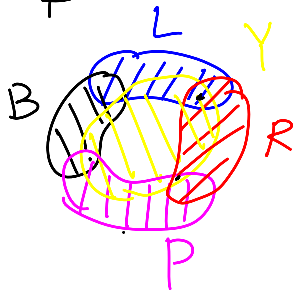
MATH 529 – Lecture 11 (02/13/2024)

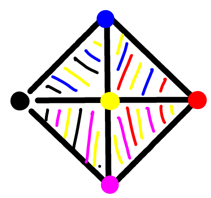
Today:
* Nerve theorem
* Čech complex
* Vietoris-Rips complex
* Delaunay complex

Recall $\mathrm{Nrv}\, F = \{ X \subseteq F \mid \cap X \neq \emptyset \}$, and the examples ...

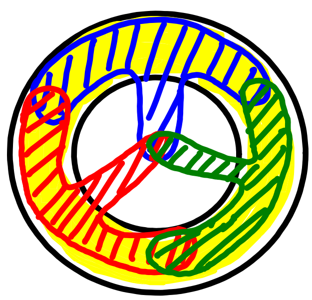
F



Nrv F



In this example, F and Nrv F are homotopy equivalent. But does this result hold in general? Let's consider another example...



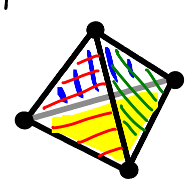
$|F|$ is a closed disc with three holes

F is a collection of four regions, such that every subset of three regions has a common intersection.

With $F = \{R, B, G, Y\}$ for Red, Blue, Green, Yellow, we can write

$$\mathrm{Nrv}\, F = \{ R, B, G, Y, \{R,B\}, \{R,G\}, \{R,Y\}, \{B,G\}, \{B,Y\}, \{G,Y\}, \{R,B,G\}, \{R,B,Y\}, \{R,G,Y\}, \{B,G,Y\} \}.$$

Nrv F



$\approx \mathbb{S}^2$

Indeed, Nrv F has a geometric realization as the surface of a tetrahedron as shown.

So $\mathrm{Nrv}\, F \not\simeq |F|$ here! $\longrightarrow$ underlying space, disk with 3 holes.

But if the sets in F are "nice", we do get homotopy equivalence with Nrv F, as specified by the following theorem.

Nerve theorem Let F be a finite collection of closed **convex** sets in $\mathbb{R}^d$. Then $\operatorname{Nrv} F$ has the same homotopy type as the collection of sets in F.

Our goal is to build simplicial complexes out of collections of points. We could consider a collection of convex sets, each containing one point from the set, and then form its nerve. A default convex set containing a point is a closed ball centered at that point. We will consider a few different ways of forming simplicial complexes out of points using balls centered on them.

Čech Complex Let S be a finite set of points in $\mathbb{R}^d$.

pronounced as "Check"

We write $B_{\bar{x}}(r) = \bar{x} + r\mathbb{B}^d = \{\bar{y} \in \mathbb{R}^d \,/\, \|\bar{y}-\bar{x}\| \leq r\}$, for the closed ball of radius r and center $\bar{x}$.

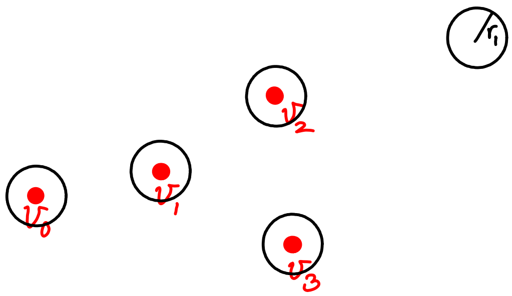
The **Čech complex** at radius r of the points in set S is the nerve of the collection of closed r-balls centered at the points.

$$\check{C}ech(r) = \{\sigma \subseteq S \mid \bigcap_{\bar{x}\in\sigma} B_{\bar{x}}(r) \neq \emptyset\}.$$

one could write $\check{C}ech_S(r)$ to be complete, but S is understood, and hence omitted, typically.

to be exact, one should say $\operatorname{conv}(\sigma)$ here. We do mean the simplex spanned by vertices in σ; $\sigma = \operatorname{conv}\{\bar{v}_0, \dots, \bar{v}_k\}$, $\bar{v}_i \in S$.

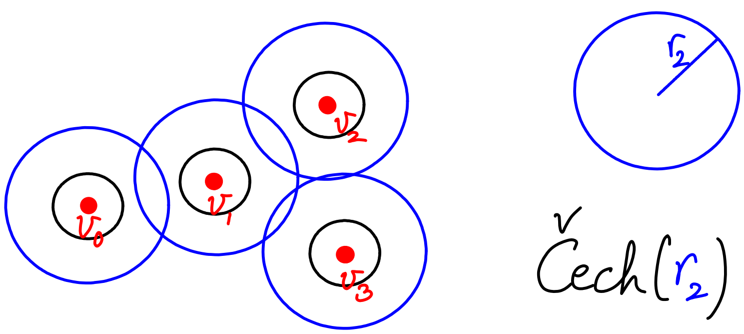
Consider an example with four points in $\mathbb{R}^2$ as shown.



$$\check{C}ech_S(r_1) \simeq \{v_0, v_1, v_2, v_3\}$$

Čech complex is homotopic to the union of balls centered at $\bar{v}_i$ — at all radii (and not just for small values such as r_1 shown here)

r_1 is small enough that no two of the balls centered at v_i intersect. Hence, Čech(r_1) has just the four points. Let's consider a bigger radius now.

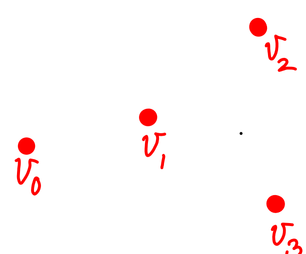


(the balls at v_1 & v_2 intersect

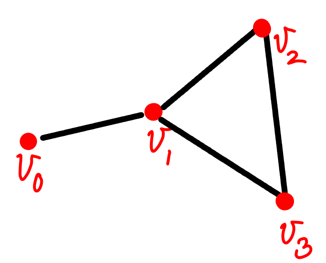
$$\check{C}ech(r_2) = \{v_0, v_1, v_2, v_3, \overline{v_0 v_1}, \overline{v_1 v_2}, \overline{v_2 v_3}, \overline{v_1 v_3}\}.$$

Geometric realizations of Čech(r_1) and Čech(r_2):

Čech(r_1):

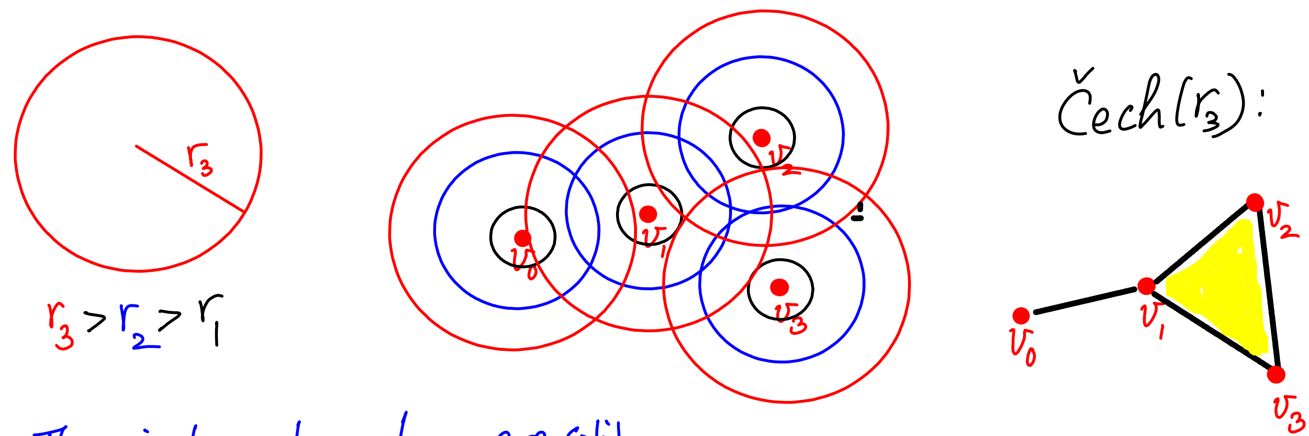


Čech(r_2) :



Notice that the balls centered at v_1, v_2, v_3 do not all intersect. Thus, there is a "hole" in between these three balls, which is represented by the empty triangle $v_1 v_2 v_3$ in Čech(r_2).

Increasing the radius a bit more brings in $\Delta v_1 v_2 v_3$:

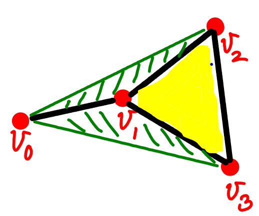


Note: The circles shown here are solid discs — shading is avoided for clarity.

Notice that Čech(r_1) is a subcomplex of Čech(r_2), which in turn is a subcomplex of Čech(r_3).

In general, **Čech(r_i) $\subseteq$ Čech(r_j)** when $r_i \le r_j$.

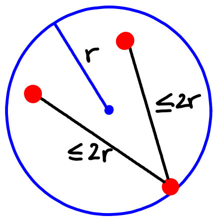
Also, Čech(r) of a set S of points in $\mathbb{R}^d$ may not have a geometric realization in $\mathbb{R}^d$ itself. But you can always treat it as an abstract simplicial complex.



At larger radii (r_4), triangles $\Delta v_0 v_1 v_2$ and $\Delta v_0 v_1 v_3$ are included in Čech(r_4), and at a still higher radius, tetrahedron $v_0 v_1 v_2 v_3$ is included. But, of course, $v_0 v_1 v_2 v_3$ cannot be embedded in $\mathbb{R}^2$.

We will consider this aspect — the complex having a geometric realization in the input space itself — later on. First, we look at more properties of the Čech complex.

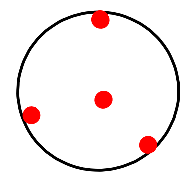
Another property central to the Čech complex is that balls of radius r have a common intersection iff their centers lie inside a ball of radius r.



So, $\sigma \subseteq S \in \check{C}ech(r) \iff$ smallest ball enclosing σ has radius $\leq r$.

Def The **miniball** of a set $\sigma \subseteq S$ is the smallest closed ball containing σ.

similar to circumsphere/circumcircle



Hence, radius of miniball of $\sigma \leq r \iff \sigma \in \check{C}ech(r)$.

To build (or define) $\check{C}ech(r)$, we need to check intersections of multiple (≥ 3) balls. This step could be computationally expensive, especially in large sets of points, since we have to go upto checking all points together in the data set! But, here is a better option.

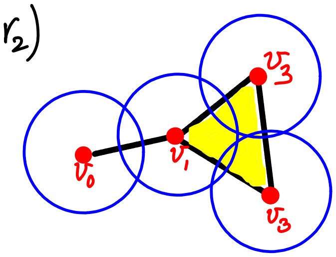
Vietoris-Rips Complexes Instead of checking the intersection of all balls, if we check just pairwise intersections, and add 2- or higher dimensional simplices whenever all edges are in, we get the **Vietoris-Rips** or VR complex.

We write $VR_S(r) = \{\sigma \subseteq S \mid \text{diam}\, \sigma \leq 2r\}$.

or $\text{Vietoris-Rips}_S(r)$ diameter of σ

Def The **diameter** of σ is the supremum of all pairwise distances between points in σ.

$VR(r_2)$



Compared to $\check{C}ech(r_2)$, we add $\Delta v_1 v_2 v_3$ to the Vietoris-Rips complex at $r = r_2$.

How do $\check{C}ech(r)$ and $VR(r)$ compare?

Naturally, $\check{C}ech_S(r) \subseteq VR_S(r)$. But notice that $VR_S(r_2)$ does not have a hole, as $\Delta v_1 v_2 v_3$ is included. At the same time, $\left|\bigcup_{i=0}^{4} B_{v_i}(r_2)\right|$ does have a hole, and so does $\check{C}ech(r_2)$.

So, homotopy is not preserved in $VR_S(r_2)$. Nonetheless, we get an inclusion going the other way, i.e., $VR(r) \subseteq \check{C}ech(r')$, at a larger radius r'.

<u>Vietoris-Rips Lemma</u> Let S be a finite set of points in $\mathbb{R}^d$, and let $r \geq 0$. Then

$$VR_S(r) \subseteq \check{C}ech(\sqrt{2}r).$$

The inclusions going both ways mean that VR and $\check{C}ech$ complexes are "quite comparable" when we consider all possible radii $(-\infty < r < \infty)$. While we may not get the same series of complexes, either family would be sufficient for most topological computations of interest. Hence, VR complexes are almost always preferred for computations, while $\check{C}ech$ complexes are sometimes preferred when used in proofs.

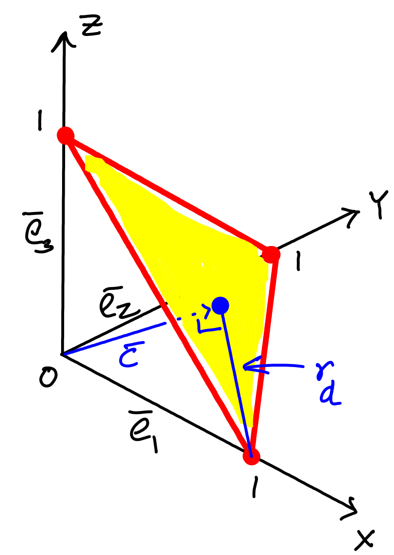
Proof (IDEA)

Consider Δ^d, the regular d-simplex in $\mathbb{R}^{d+1}$. Each vertex is a unit vector in this space. Thus,

$$\Delta^d = \mathrm{conv}(\bar{e}_1, \dots, \bar{e}_{d+1}), \text{ where } \bar{e}_j \text{ is the } j^{th} \text{ unit vector in } \mathbb{R}^{d+1}.$$

Regular simplices are the "limiting" cases to consider here, due to their symmetry.

Let $\bar{c}$ be the barycenter of Δ^d.



$$\bar{c} = \begin{bmatrix} \frac{1}{d+1} \\ \vdots \\ \frac{1}{d+1} \end{bmatrix} \qquad \|\bar{c}\| = \frac{1}{\sqrt{d+1}} \text{ is the length from origin of } \Delta^d.$$

$\hookrightarrow$ perpendicular distance

We compute $r_d = \sqrt{\dfrac{d}{d+1}} \left(= \sqrt{1 - \|\bar{c}\|^2} \right)$.

Note: $r_d \longrightarrow 1$ as $d \to \infty$.

The pairwise distance between $\bar{e}_i$ and $\bar{e}_j$ in σ is $\sqrt{2}$.

Also, the miniball of Δ^d has radius r_d.

Hence, simplex Δ^d of diameter $\sqrt{2}$ also belongs to $\check{C}ech(r_d)$. Multiplying by $\sqrt{2}\,r$, we get that,

$$VR(r) \subseteq \check{C}ech(\sqrt{2}\, r\, r_d). \quad \text{But } r_d \leq 1,$$

and hence $\quad VR(r) \subseteq \check{C}ech(\sqrt{2}\, r)$. $\square$

We saw $\check{C}ech_S(r)$ and $VR_S(r)$ of S (points) in $\mathbb{R}^d$.

Can we limit the dimension of the simplices we get from S? Yes!

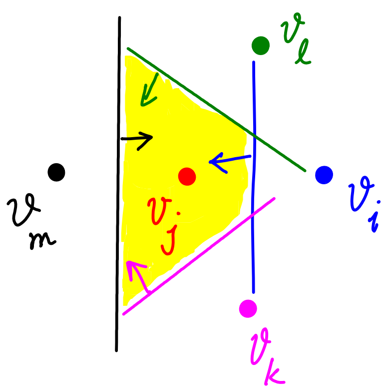
We can build the Delaunay complex. We first describe its dual construction.

Voronoi Diagram

Recall: $S = \{\bar{v}_1, \ldots, \bar{v}_n\}$ is a finite set of points in $\mathbb{R}^d$.

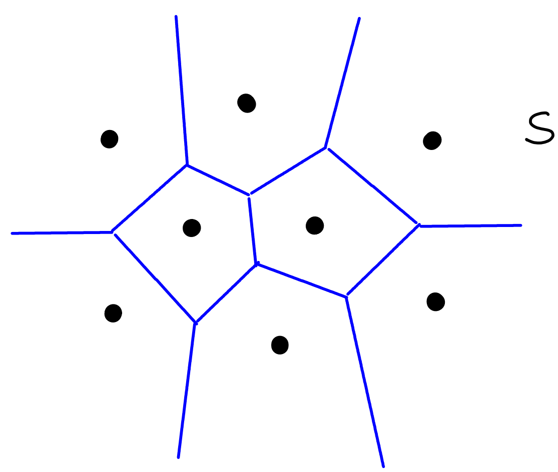
The Voronoi cell of $\bar{v}_j \in S$ is the set of points in $\mathbb{R}^d$ closest to $\bar{v}_j$:

$$V_{\bar{v}_j} = \{\bar{x} \in \mathbb{R}^d \mid \|\bar{x} - \bar{v}_j\| \leq \|\bar{x} - \bar{v}_i\| \; \forall \, \bar{v}_i \in S\}.$$



When we have just two points, say, v_i and v_j, the perpendicular bisector between them is the set of points equidistant from both of them. The half plane on the side of v_j then is V_{v_j}, its Voronoi cell.

$V_{\bar{v}_j}$ is a convex polyhedron, as it is the intersection of a set of half spaces, each being convex. $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ together tile or cover all of $\mathbb{R}^d$.



The collection of $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ is called the **Voronoi diagram** of S.

$V_{\bar{v}_i}$ and $V_{\bar{v}_j}$ meet at most in a common boundary. In $\mathbb{R}^2$, Voronoi cells meet at points or edges.

Notice that $V_{\bar{v}_j}$ can be open or closed. Intuitively, the boundary of $V_{\bar{v}_j}$ can be thought of as the "fence" around $\bar{v}_j$'s "house" – everything within the fence "belongs" to $\bar{v}_j$.

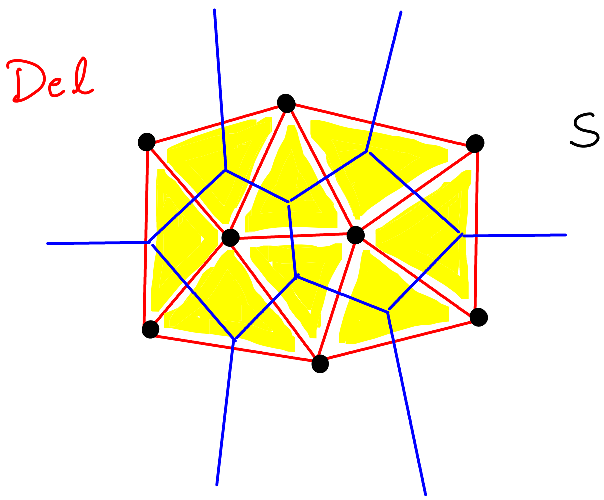
Delaunay Triangulation

The **Delaunay complex** of S is (isomorphic to) the nerve of its Voronoi diagram.

$$\mathrm{Del}_S = \left\{ \sigma \subseteq S \,\middle|\, \bigcap_{\bar{v}_j \in \sigma} V_{\bar{v}_j} \neq \phi \right\}.$$

or $\mathrm{Delaunay}_S$

Similar to the Čech complex, we start with a convex set or cell associated with each point in S, and then take the nerve. But instead of balls, we use the Voronoi cells for each vertex.



Shown here is one geometric realization of $\mathrm{Delaunay}_S$. This is the "natural" realization, as well.