MATH 529 — Lecture 14 (02/22/2024)

Today: * more on groups, homomorphisms
* chains and chain groups on simplicial complexes

Recall group G with operation $*$ ...

__Induced operation__ Let $\langle G, *\rangle$ be a group, and $S \subseteq G$. If S is closed under $*$, then $*$ is the **induced operation** on S from G.

__Subgroup__ $H \subseteq G$ (subset) is a **subgroup** of $\langle G, *\rangle$ if H is a group and is closed under $*$.

$\{e\}$ is the trivial subgroup of G. All other subgroups are nontrivial.

__**Theorem**__ $H \subseteq G$ of a group $\langle G, *\rangle$ is a subgroup of G if and only if

(a) H is closed under $*$;

(b) identity e of G is in H; and

(c) $\forall a \in H$, $a^{-1} \in H$.

This theorem could also be used as the definition of a subgroup.

e.g., Consider $\mathbb{Z}_4$

$H = \{0, 2\}$ is a subgroup.

Notice that $2+2 = 0 \pmod 4$, and hence H is indeed closed under the operation in question (addition modulo 4).

$\mathbb{Z}_4$	$\boxed{0}$	1	$\boxed{2}$	3
$\boxed{0}$	$\boxed{0}$	1	2	3
1	1	2	3	0
$\boxed{2}$	$\boxed{2}$	3	$\boxed{0}$	1
3	3	0	1	2

$H = \{0, 2\}$ is the only nontrivial subgroup of $\mathbb{Z}_4$ (distinct from $\mathbb{Z}_4$ itself).

<u>Cosets</u> Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by $a \sim_L b$ iff $\underbrace{a^{-1}b}_{a^{-1} * b} \in H$. Similarly, $\sim_R$ is defined on G by $a \sim_R b$ iff $ab^{-1} \in H$. Note that $\sim_L$ and $\sim_R$ are equivalence relations on G. Also

$$a^{-1}b \in H \text{ or } a^{-1}b = h \in H \implies b = ah.$$

For $a \in G$, the subset $aH = \{ah \mid h \in H\}$ of G is the **left coset** of H containing a, and $Ha = \{ha \mid h \in H\}$ is the **right coset** of H containing a.

If G is abelian, then $ah = ha$ $\forall a \in G, h \in H$. Then the left and right cosets match, i.e., $aH = Ha$.

e.g., $\mathbb{Z}_4$, $H = \{0, 2\}$ is a subgroup. The coset of 1 is $1 +_4 \{0,2\} = \{1,3\}$.

Our goal is to use groups to characterize topological spaces. Hence, we need to be able to characterize the "structure" of groups. We could use maps between groups for this purpose.

To simplify notation, we write ab for $a * b$, with $*$ understood.

<u>Homomorphisms</u> A map $\varphi : G \to G'$ is a ***homomorphism*** if $\underbrace{\varphi(ab)}_{} = \varphi(a)\varphi(b)$ $\forall a, b \in G$.

$$\varphi(a *_G b) = \varphi(a) *_{G'} \varphi(b)$$

We can always define a trivial homomorphism by setting $\varphi(g) = e' \ \forall g \in G$, where e' is the identity of G'.

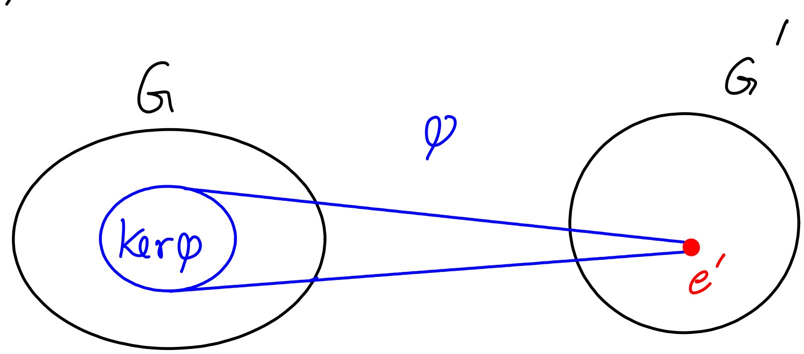
Homomorphisms preserve identity, inverse, and subgroups.

Theorem Let $\varphi: G \to G'$ be a homomorphism. Then

1. $\varphi(e) = e'$, where e, e' are identities of G, G', respectively.
2. $\varphi(a^{-1}) = (\varphi(a))^{-1} \ \forall a \in G$.
3. $H \subseteq G$ is a subgroup of $G \Rightarrow \varphi(H)$ is a subgroup of G'.
4. $K' \subseteq G'$ is a subgroup of $G' \Rightarrow \varphi^{-1}(K')$ is a subgroup of G.

This theorem could be used, alternatively, as a definition of homomorphism.

Kernel Let $\varphi: G \to G'$ be a hom'm. The subgroup $\varphi^{-1}(\{e'\}) \subseteq G$ is the **kernel** of φ.



Notice that $\{e'\}$ is the trivial subgroup of G'. Hence by (4) of the Theorem above, $\ker\varphi$ is a subgroup of G.

Since $\ker\varphi$ is a subgroup of G, we can define kernel cosets. Let $H = \ker\varphi$, $a \in G$, then

$$aH = \varphi^{-1}\{\varphi(a)\} = \{x \in G \mid \varphi(x) = \varphi(a)\} = Ha.$$

Intuitively, any $h \in \ker\varphi$ gets mapped to the identity (e'). So, we could add h to a to get x, and x also gets mapped to the image of a.

If $gH = Hg$ $\forall g \in G$ for a subgroup H of G, then H is a **normal** subgroup of G. All subgroups of Abelian groups are normal, and so is $\ker\varphi$ for a homomorphism φ.

The properties of a function being injective, surjective, or both can be studied for homomorphisms as well. But for homomorphisms, we refer to these properties using terms specific to groups.

Maps in general	homomorphisms between groups
1-1	monomorphism
onto	epimorphism
1-1 and onto (bijection)	isomorphism ($\cong$) notation

Isomorphism between groups is like homeomorphism between topological spaces. Recall previous discussion about ASCs being isomorphic.

<u>Finitely generated</u> Let $a_i \in G$ for $i \in I$, an index set. The smallest subgroup of G containing $\{a_i \mid i \in I\}$ is the subgroup generated by $\{a_i \mid i \in I\}$. If this subgroup is all of G, then $\{a_i \mid i \in I\}$ **generates** G, and a_i are the generators. If I is finite, then G is **finitely generated**.

For instance, both $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}_4, +_4 \rangle$ are finitely generated.

Homology Groups

We will talk about homology for simplicial complexes, i.e., simplicial homology.
Homology studies "holes" in spaces, i.e., holes in 1D, enclosed voids in 2D, etc. Interestingly, holes are characterized by what surrounds them!

Let K be a simplicial complex. A **p-chain** (for $p \leq \dim K$) of K is a formal sum of the p-simplices of K.
$\rightarrow$ "linear combination"

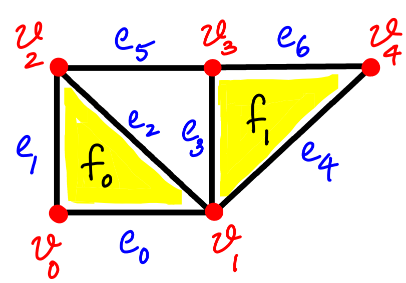
$$\bar{c} = \sum_{i=1}^{m} a_i \sigma_i \; ,$$

K has m p-simplices, $\sigma_1, \ldots, \sigma_m$, and a_i's are their coefficients.

To define groups using addition modulo 2, i.e., in $\mathbb{Z}_2$, $a_i \in \{0,1\}$.
When using addition over $\mathbb{Z}$, $a_i \in \mathbb{Z}$. We could define homology groups over $\mathbb{Z}_2, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$, or over any ring.
$\rightarrow$ rational numbers

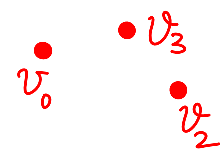
Examples

K:



0-chain (over $\mathbb{Z}_2$)

$\bar{c}_0$:



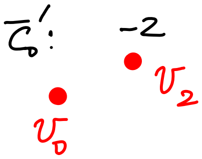
$a_i = 1 \quad i = 0,2,3$

$a_i = 0, \quad i = 1,4$

$$\bar{c}_0 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

0-chain (over $\mathbb{Z}$)

$\bar{c}_0'$:

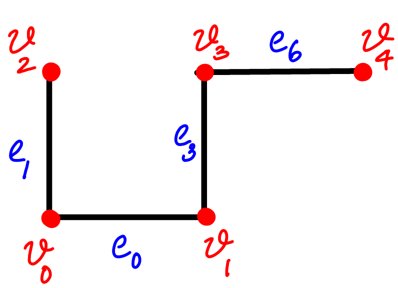


$$\bar{c}_0' = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

1-chains

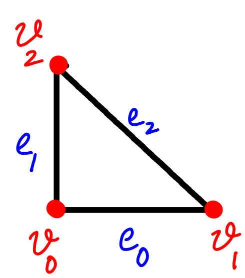
over $\mathbb{Z}_2$:

$\bar{c}_1$:

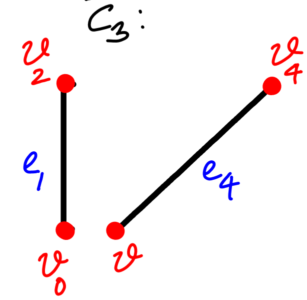


$$\bar{c}_1 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

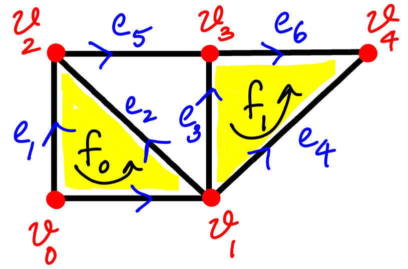
$\bar{c}_2$:



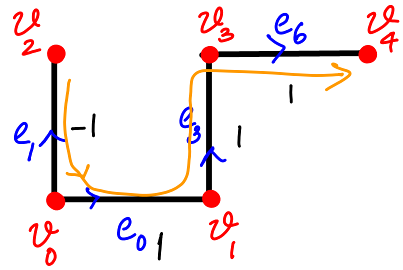
$\bar{c}_3$:



Note that the chain need not be connected.

K: 

We consider orientations over $\mathbb{Z}$.

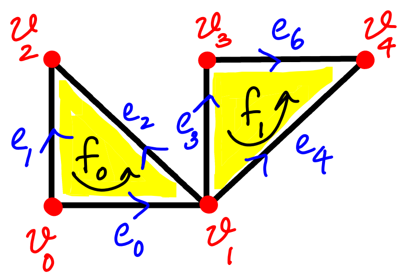
$\bar{c}_1'$: 

$$\bar{c}_1' = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We could have an "overall orientation" for the 1-chain, but this is not needed always.

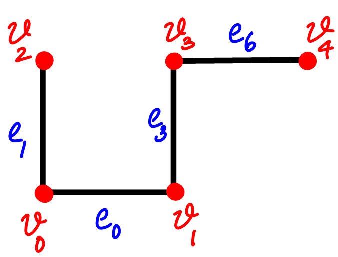
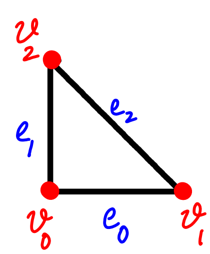
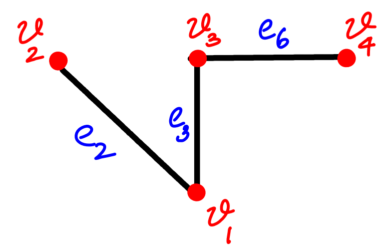
2-chains

$\bar{d}_1 = \begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 2-chain (over $\mathbb{Z}_2$ or $\mathbb{Z}$)



$\bar{d}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a 2-chain over $\mathbb{Z}$.

We can add two p-chains component- or p-simplex-wise, i.e., by adding their vectors. If $\bar{c} = \sum_{i=1}^{m} a_i \sigma_i$, $\bar{c}' = \sum_{i=1}^{m} b_i \sigma_i$, then $\bar{c} +_2 \bar{c}' = \sum_{i=1}^{m} (a_i +_2 b_i)\sigma_i$

$\bar{c}_1$:  $+_2$ $\bar{c}_2$:  $=$ $\bar{c}_4$: 

Equivalently, we add the corresponding vectors:

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}}_{\bar{c}_1} +_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}}_{\bar{c}_2} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}}_{\bar{c}_4}$$

p-chains with addition ($+_2$ over $\mathbb{Z}_2$ or $+$ over $\mathbb{Z}$) form the group of p-chains of K, denoted $\langle C_p(K), +_2\rangle$ or $\langle C_p(K), \mathbb{Z}\rangle$, or simply $C_p(K)$ (or just C_p).

$C_p(K)$ is indeed a group, and is an Abelian group.

* identity: $\bar{0} = \sum_{i=1}^{m} 0\sigma_i$
* inverse: inverse of $\bar{c}$ is $-\bar{c}$ over $\mathbb{Z}$, and $\bar{c}$ over $\mathbb{Z}_2$ (as $\bar{c} +_2 \bar{c} = \bar{0}$).
* $+_2$ and $+$ are associative
* $+_2$ and $+$ are commutative.

For K, there is $C_p(K)$ for $0 \le p \le \dim(K)$.

If K has m p-simplices, each p-chain can be represented by an m-vector. The p-chains corresponding to the m unit vectors are the **elementary p-chains** of K, i.e., they correspond to each σ_i.

$$\bar{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \to \sigma_i \\ \\ \\ \\ \end{matrix}$$

When K is finite, these m elementary chains generate $C_p(K)$, making it finitely generated.

How are $C_p(K)$ related for various p?

We use "boundary", defined as homomorphisms between $C_p(K)$ and $C_{p-1}(K)$ to connect these chain groups.

We first define the boundary of a single simplex, i.e., an elementary chain, and then extend it naturally to chains. We provide the definition over both $\mathbb{Z}_2$ and $\mathbb{Z}$.

<u>Boundary</u> The **boundary** of a p-simplex is the sum of all its $(p-1)$-faces.

same notation used for orientation, which we do not use over $\mathbb{Z}_2$.

If $\sigma = \text{conv}\{v_0, \dots, v_p\}$, or $\sigma = [v_0 v_1 \dots v_p]$, then

p-boundary $\partial_p \sigma = \sum_{j=0}^{p} [v_0 \dots \hat{v}_j \dots v_p]$, where $\hat{v}_j$ means v_j is omitted.

Over $\mathbb{Z}$, we have $\partial_p \sigma = \sum_{j=0}^{p} (-1)^j [v_0, \dots, \hat{v}_j, \dots v_p]$, which is the sum of all its $(p-1)$-faces with their induced orientations.

Notice that $\partial_p \sigma$ is a $(p-1)$-chain (in both cases).