MATH 529 – Lecture 15 (02/27/2024)

Today: * boundary of chains
* cycles, boundaries, homology groups

Recall boundary of p-simplex σ (over $\mathbb{Z}_2$ or $\mathbb{Z}$).

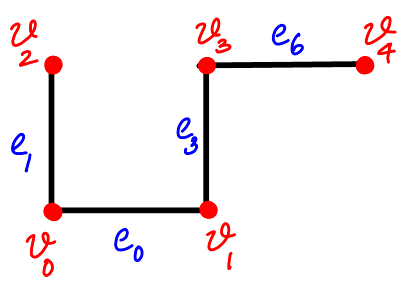
Note that $\partial_p \sigma$ is a $(p-1)$-chain (in both cases).

The boundary of a p-chain is the sum of the boundaries of its p-simplices.

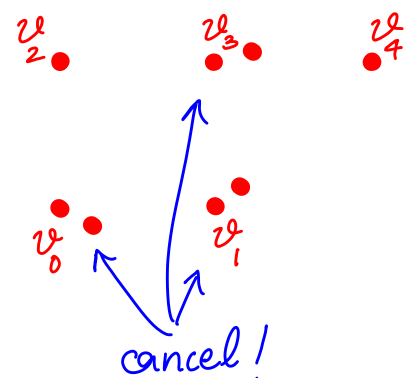
$$\bar{c} = \sum_{i=1}^{m_1} a_i \sigma_i \implies \partial_p \bar{c} = \sum_{i=1}^{m_1} a_i (\partial_p \sigma_i),$$ which is also a $(p-1)$-chain.

Examples

$\bar{c}_1$:



$\implies \partial_1(\bar{c}_1) =$



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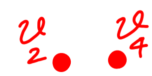
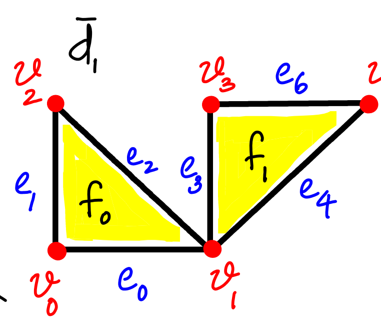
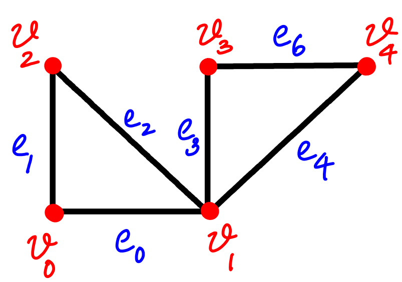


Note that vertices shared by two edges in the chain do not appear in its boundary.

$\partial_2\Big($

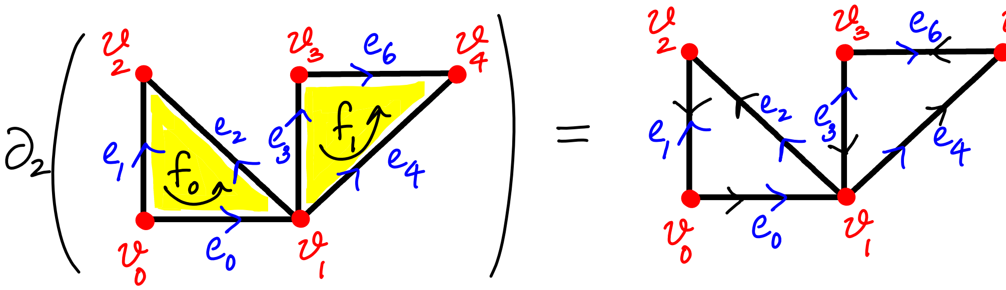


$\Big) =$



The boundary of a triangle is made of its edges.

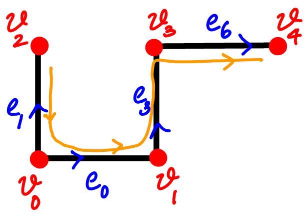
Consider $\partial_2 \bar{d}_1$ over $\mathbb{Z}$ now:



$$\partial_2 \bar{d}_1 = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Notice that the induced orientation on $\bar{e}_1$ is opposite to its own orientation.

$\bar{c}_1$:



$$\partial \bar{c}_1 = -(v_2 - \cancel{v_0}) + (\cancel{v_1} - \cancel{v_0}) + (\cancel{v_3} - \cancel{v_1}) + (v_4 - \cancel{v_3}) = v_4 - v_2.$$

The dimension is often omitted, and we just talk about $\partial \bar{c}$ of a p-chain $\bar{c}$, with the dimension understood.

Taking the boundary maps a p-chain to a $(p-1)$-chain. Equivalently, we can talk about the map $\partial_p : C_p \to C_{p-1}$. Notice that such a map is defined for each p in the range $1 \le p \le \dim K$.

Also, $\underbrace{\partial_p(\bar{c} +_2 \bar{c}') = \partial_p \bar{c} +_2 \partial_p \bar{c}'}$ for 2 p-chains $\bar{c}, \bar{c}'$. Hence ∂_p is a homomorphism, referred to as the **p-th boundary map** or *homomorphism*.

commutes with $+_2$, which is the definition of a homomorphism

Similarly, we have ∂_{p-1}, which is the $(p-1)$-st boundary homomorphism, and ∂_{p-2}, and so on. $\partial_{p-1} : C_{p-1} \to C_{p-2}$.

∂_p is a homomorphism over $\mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}$) as well, not just over $\mathbb{Z}_2$.

A **chain complex** is a sequence of chain groups connected by boundary homomorphisms

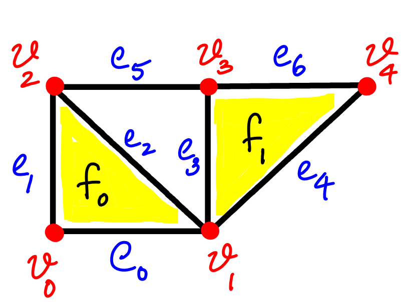
$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \rightarrow \cdots$$

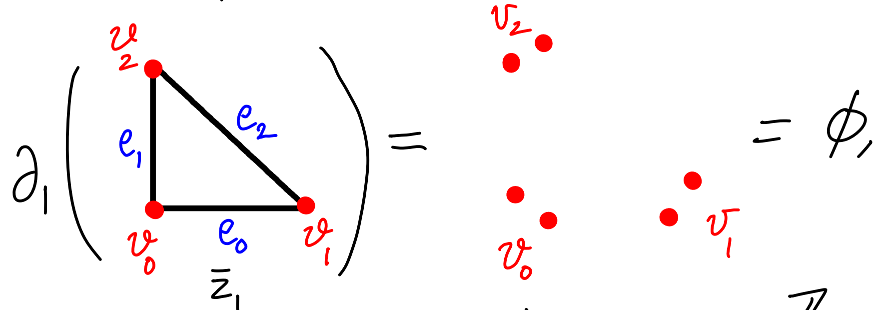
The word complex as used here is different from a simplicial complex. At the same time, the chain complex is indeed an abstract simplicial complex, with elements connected by boundary homomorphisms.

Cycles

A **p-cycle** is a p-chain with empty boundary

For instance, consider the 1-chain $\bar{z}_1$ of K:

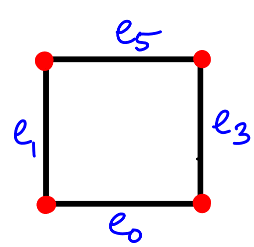
K: 



with addition over $\mathbb{Z}_2$.

Hence $\bar{z}_1$ is a 1-cycle.

Alternatively, a p-chain $\bar{c}$ is a p-cycle if $\partial \bar{c} = 0$.

Here is another 1-cycle : $\bar{z}_2$: 

Since ∂ commutes with $+$, the p-cycles of K form a group, denoted by Z_p or $Z_p(K)$. Z_p is a subgroup of C_p. Also, $Z_p = \ker \partial_p$, i.e., Z_p is the kernel of the p^{th} boundary homomorphism. Notice that $\partial_p \bar{z} = 0$ for $\bar{z} \in Z_p$. Also, just as C_p is, Z_p is an Abelian group.

The surface of a tetrahedron (made of four triangles) is a 2-cycle. Notice that its boundary is empty.

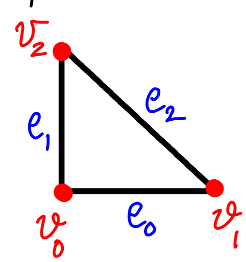
So, all cycles are also chains, but not the other way usually. But for $p=0$, $\partial_0 v_j = 0$, i.e., the boundary of a vertex is empty (by definition). Hence every 0-chain is also a 0-cycle, i.e., $Z_0 = C_0$.

But typically, $Z_p \subset C_p$ for $p \geq 1$.

<u>Boundaries</u> A **p-boundary** $\bar{b}$ is a p-chain that is the boundary of some $(p+1)$-chain $\bar{d}$, i.e., $\bar{b} = \partial_{p+1} \bar{d}$ for $\bar{d} \in C_{p+1}$.

Again, since ∂ commutes with $+_2$ (or $+$) we get a group of p-boundaries B_p (or $B_p(K)$). B_p is a subgroup of Z_p, and of C_p. B_p is the image of $\partial_{p+1} : C_{p+1} \to C_p$. $B_p = \operatorname{im} \partial_{p+1}$. B_p is abelian.

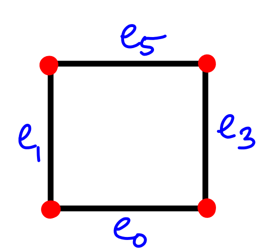
$\bar{z}_1$



$\bar{z}_1 = \partial \bar{f}_0$, and hence is a 1-boundary.

2-chain made of triangle f_0

$\bar{z}_2$:



But $\bar{z}_2$ is not a 1-boundary.

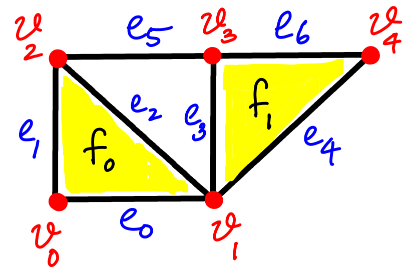
We have $B_p \subset Z_p \subset C_p$ (in general).

So, in summary, p-cycle $\bar{z}$: $\partial \bar{z} = 0$; $\quad Z_p = \ker \partial_p$.

p-boundary $\bar{b}$: $\bar{b} = \partial_{p+1} \bar{d}$; $\quad B_p = \text{im } \partial_{p+1}$.

Examples

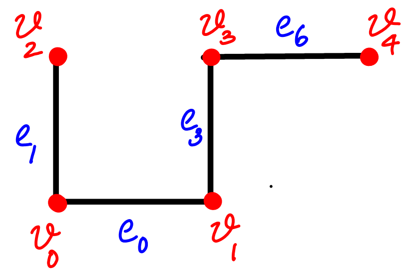
K_1:



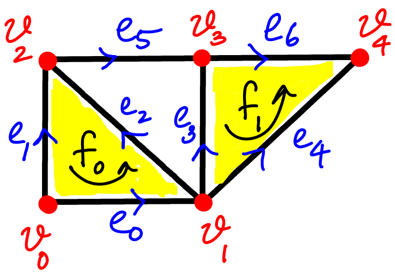
is a 0-boundary,

as it is $\partial_1 \bar{c}_1$.

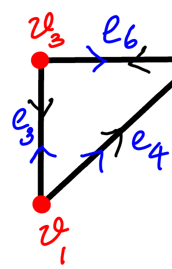
$\bar{c}_1$:



Now consider oriented K_1 (over $\mathbb{Z}$):



$\bar{b}_1$:



$$\bar{b} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

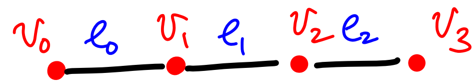
$\bar{b}_1$ is a 1-boundary, as $\bar{b}_1 = \partial_2 \bar{f}_1 \rightarrow$ elementary 2-chain of f_1

Let's try to enumerate how many p-cycles and p-boundaries are there for general p. We want to study cycles that are not boundaries.

p=0 case (over $\mathbb{Z}_2$)

$\bar{c}$, a 1-chain, is a collection of edges. $\partial_1 \bar{c}$ gives end points of edges with duplicate end points canceled in pairs, leaving an even number of distinct v_j's.

If K is connected, for every set of even number of vertices, we can find paths made of edges that connect the vertices such that the 1-chain made of these edges has as its boundary the collection of vertices.



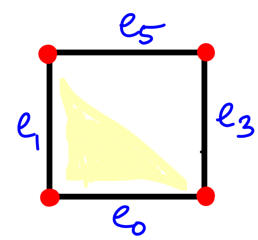
e.g., $\partial_1(e_0+e_2) = \{v_0, v_1, v_2, v_3\}$.

Thus, every set of even number of vertices (v_j's) is a 0-boundary, while every odd set is not. Hence, if K is connected, exactly half of the 0-cycles are 0-boundaries.

But we typically cannot make similar statements about counts of p-cycles and p-boundaries for $p \geq 1$.

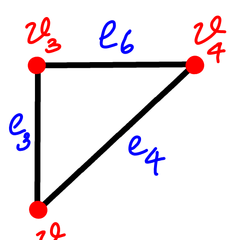
We want to characterize cycles that are not boundaries, as they capture holes, e.g., as $\bar{z}_2$ here.

$\bar{z}_2$:



<u>An observation:</u> Consider $\bar{b}_1 = \partial_2 \bar{f}_1$.

$\bar{b}_1$:



$\partial \bar{b}_1 = 0$ (as each v_j cancels in pairs).

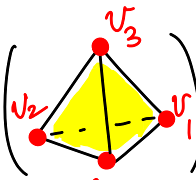
In other words, $\partial_1 \partial_2 f_1 = 0$. This result holds in general!

<u>**Fundamental Lemma of Homology**</u> $\partial_p \partial_{p+1} \bar{d} = 0 \quad \forall p \in \mathbb{Z}$.

In words, boundary of a boundary is empty.

<u>Proof (over $\mathbb{Z}_2$)</u> For each $(p+1)$-simplex τ, $\partial_p \partial_{p+1} \tau = 0$, as $\partial_{p+1}\tau$ consists of all p-faces of τ. Each $(p-1)$-face of τ belongs to exactly two p-faces.

$$\partial_2\Big(\partial_3\Big(\text{[tetrahedron]}\Big)\Big) = \partial_2\big(\text{union of 4 triangles}\big) = 0,$$



tetrahedron

as each edge is shared by exactly two triangles.

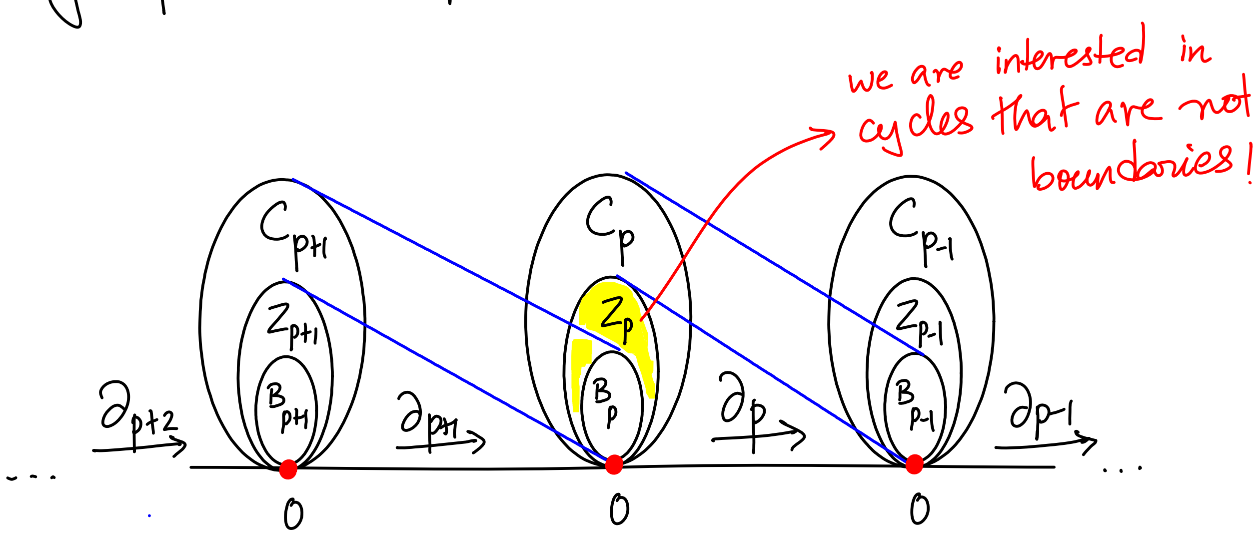
$$\partial_p\big(\partial_{p+1}\tau = \partial_{p+1}([v_0 \ldots v_{p+1}])\big) = \partial_p\Big(\sum_{j=0}^{p+1} [v_0 \ldots \hat{v}_j \ldots v_{p+1}]\Big)$$

WLOG, the $(p-1)$-simplex $[v_0 \ldots v_{p-1}]$ is a face of $[v_0 \ldots v_{p-1} v_p]$ and $[v_0 \ldots v_{p-1} v_{p+1}]$.

The result holds over $\mathbb{Z}$ as well (over any ring, in fact). We must consider induced orientations when taking ∂_{p+1} and ∂_p.

A p-boundary is also a p-cycle. Hence B_p is a subgroup of Z_p.

The groups C_p, Z_p, B_p for various p are related as follows:



Homology Groups

Since B_p is a subgroup of Z_p, we can take quotients.

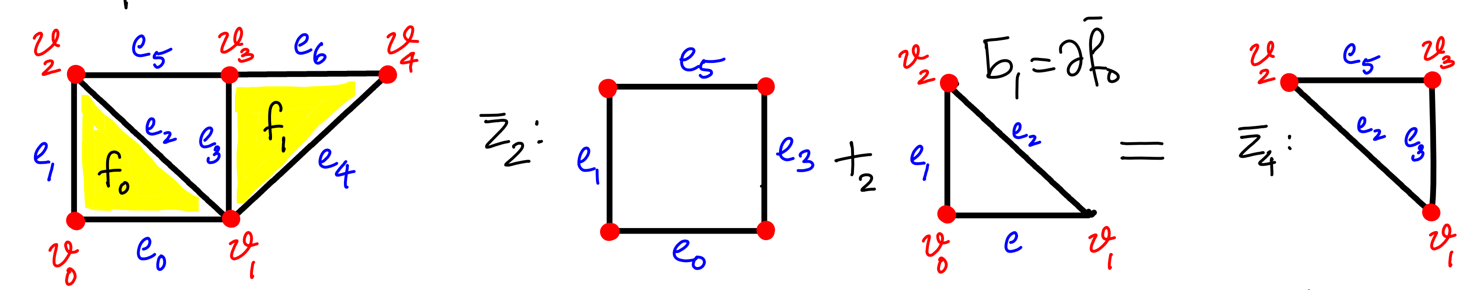
Def The **p-th homology group** is the p-th cycle group modulo the p-boundary group, $H_p = Z_p / B_p$.

H_p has the classes of cycles that are not boundaries.

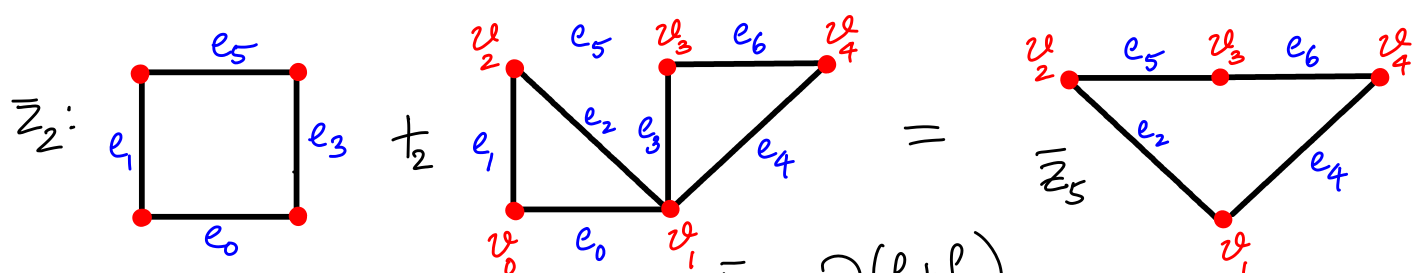
Each element of $H_p(K)$ is obtained by adding p-boundaries to given p-cycles (in their class), i.e., $\bar{z} + B_p$, where $\bar{z} \in Z_p$.

$\bar{z} + B_p$ is a coset of B_p in Z_p.

Example



$\bar{z}_2$ and $\bar{z}_4$ are cycles going around the same hole.



$\bar{b}_3 = \partial_2(f_0 + f_1)$

$\bar{z}_5$ also goes around the same hole as $\bar{z}_2$ (and $\bar{z}_4$).

We could use any one cycle going around the hole $(\bar{z}_2, \bar{z}_4, \bar{z}_5)$ as a representative of $H_1(K)$.

In general, another cycle $\bar{z}' = \bar{z} + \bar{b}$ for $\bar{z}$ in $H_p(K)$ and $\bar{b} \in B_p$ is in the same class as $\bar{z}$, i.e., $\bar{z}' + B_p = \bar{z} + B_p$ (as $\bar{b} + B_p = B_p$ itself). This is a class in $H_p(K)$, and any two cycles in this class are said to be **homologous**, written as $\bar{z} \sim \bar{z}'$.

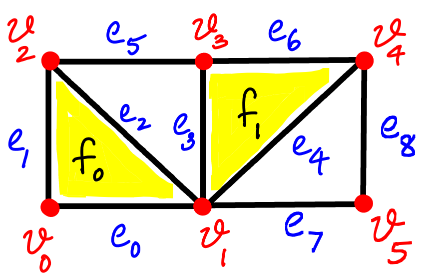
In this setting, addition of classes is well-defined:

$(\bar{z} + B_p) + (\bar{z}' + B_p) = (\bar{z} + \bar{z}' + B_p)$, independent of the particular representatives $\bar{z}$ and $\bar{z}'$.

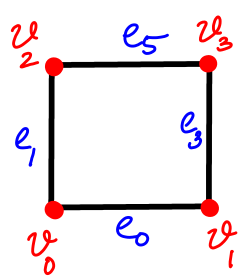
Thus H_p is indeed a group, and is abelian.

<u>Another example</u>

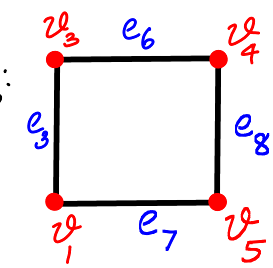
K_2:



$\bar{z}_2$:



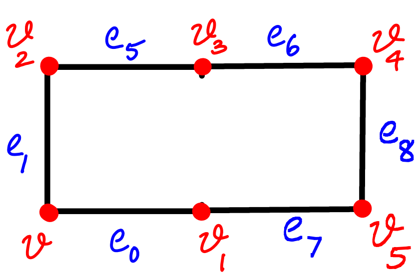
$\bar{z}_6$:



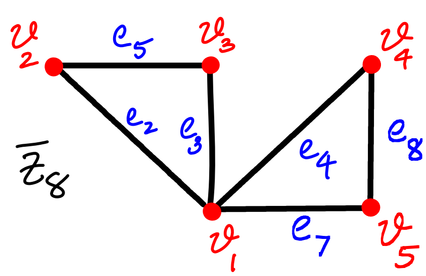
Notice $\bar{z}_6 \nsim \bar{z}_2$, as we cannot get $\bar{z}_6 = \bar{z}_2 + \bar{b}$ for any $\bar{b} \in B_1$.

Now consider two more cycles $\bar{z}_7$ and $\bar{z}_8$ as shown.

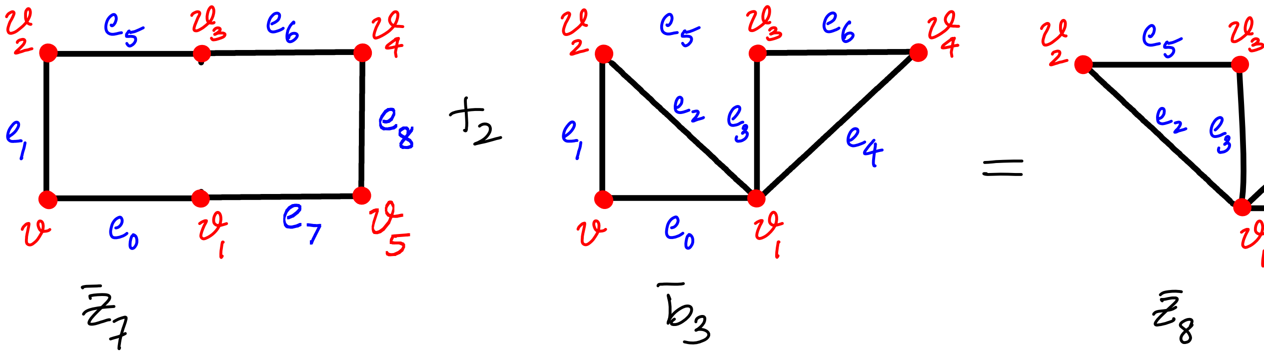
$\bar{z}_7$:



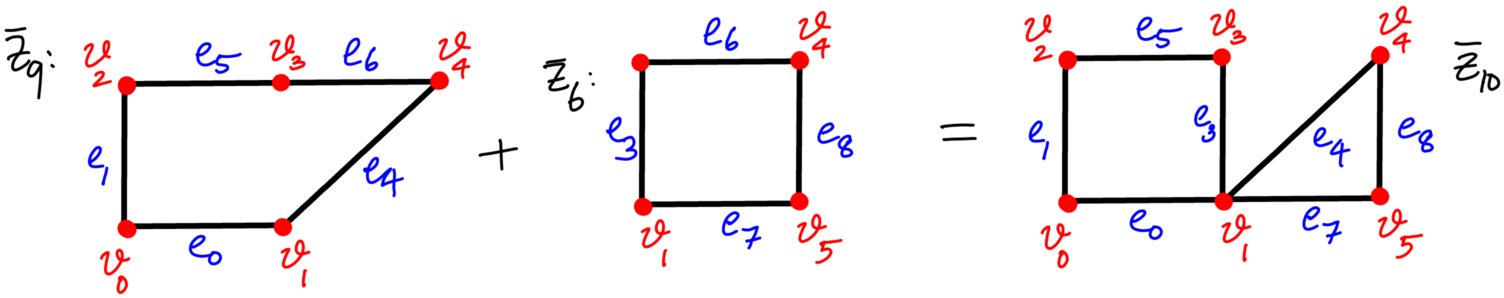
$\bar{z}_8$



$\bar{z}_7 \sim \bar{z}_8$ as



Now, consider $\bar{z}_9$ and $\bar{z}_6$. Notice that $\bar{z}_9 \nsim \bar{z}_6$. But



$\bar{z}_{10} = \bar{z}_9 + \bar{z}_6$. $\bar{z}_{10} \nsim \bar{z}_9$, but $\bar{z}_{10} \sim \bar{z}_8$

homology class of $\bar{z}_2$

To describe $H_1(K_2)$ completely, we could present $[\bar{z}_2]$ and $[\bar{z}_6]$, or equivalently, $[\bar{z}_2]$ and $[\bar{z}_7]$, or $[\bar{z}_6]$ and $[\bar{z}_7]$.

Intuitively, since K_2 has two holes, we expect $H_1(K_2)$ to have two classes.