# MATH 529 – Lecture 18 (03/07/2024)

Today: * Reduction algorithm for $[\partial_p]$
* reduced homology
* relative homology

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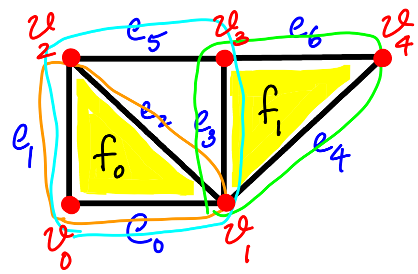
A final observation on the matrix reduction example...

K: 


$$\mathrm{SNF}([\partial_1]) = \begin{array}{c} \\ v_0 \\ v_{01} \\ v_{012} \\ v_{0123} \\ v_{01234} \end{array} \begin{array}{c} \begin{array}{ccccccc} e_0 & e_{01} & e_{013} & e_{34} & e_{012} & e_{0135} & e_{346} \end{array} \\ \left[\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & 0 & 0 & 0 & & \\ & 0 & 1 & 0 & 0 & 0 & \\ & & 0 & 1 & & 0 & 0 \\ \hline & & & 0 & & & 0 \end{array}\right] \end{array}$$

The labels for the $Z_1$ columns in $\mathrm{SNF}([\partial_1])$: $e_{012}$ $e_{0135}$ $e_{346}$

The column labels of the last three (zero) columns of $\mathrm{SNF}([\partial_1])$ gives a basis for $Z_1$. They are $e_{012}$, $e_{0135}$, and $e_{346}$, i.e., $\{\{e_0, e_1, e_2\}, \{e_0, e_1, e_3, e_5\}, \{e_3, e_4, e_6\}\}$. These three loops consist of $\partial f_0$, $\partial f_1$, and $\{e_0, e_1, e_3, e_5\}$, which captures the hole. $\{e_0, e_1, e_3, e_5\}$ can be used to represent $H_1$ here.

Notice that we may not obtain the actual "boundary of the hole" – we're guaranteed to get one loop around the hole, but it may not be the shortest or tightest loop!

# Algorithm for SNF over $\mathbb{Z}_2$ of $B \in \{0,1\}^{m \times n}$

Here is the main block:

```
void REDUCE(i)
   if ∃ j≥i, k≥i with B_jk = 1 then
       R_j ⇄ R_i ; C_k ⇄ C_i
       for h = i+1 to m
           if B_hi = 1 then
               R_h + R_i ;
           endif
       endfor
       for l = i+1 to n
           if B_il = 1 then
               C_l + C_i ;
           endif
       endfor
       REDUCE(i+1);
   end if
```

$B_{ii} = 1$ after this step (this step allows the possibility of $B_{ii} = 1$ to start with)

Zero out 1's below $B_{ii}$ using replacement EROs

zero out 1's to the right of $B_{ii}$ using replacement ECOs

proceed to next smaller block to right and below

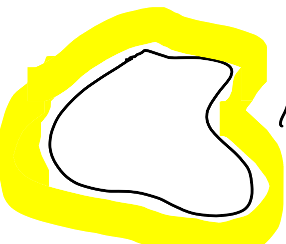
We could initiate (as identity matrices) the elementary matrices $U_{m\times m}$ and $V_{n\times n}$, as well as $U^{-1}$, and update them along the way to identify bases for $Z_p$, $B_{p-1}$, etc.

You will get the chance to implement this reduction algorithm in HW5...

# Reduced Homology groups

To take care of the discrepancy at $p=0$ ($z_0 = \mathcal{B}_0$ or not?)

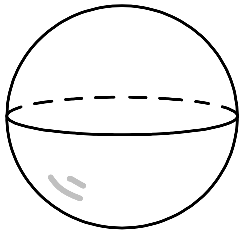
IDEA: "A prison cell in $p$-dimensions has $(p-1)$-dimensional walls".



hole

$p=2, \beta_1 = 1$

2D prison cell with 1D wall

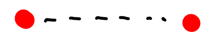


sphere (hollow)

$p=3, \beta_2 = 1$

3D prison cell with 2D wall

We count $\beta_1 = 1$ for the hole when the 2D solid patch is missing in the middle. Similarly, when the solid 3D object/space is missing from the middle of a sphere, we set $\beta_2 = 1$.



$\beta_0 = 1$?

Note that we're removing a $p$-ball in each case to make the prison cell (2-ball $\approx$ disc, 3-ball $\approx$ solid 3D patch, 1-ball $\approx$ edge).

Under this idea, we would count $\beta_0 = 1$ when we have two vertices (with the "edge in between missing", capturing a "hole"). But if $\beta_0$ is to count the number of connected components, $\beta_0 = 2$ here.

To resolve this confusion, we add the augmentation map $\epsilon : C_0 \to \mathbb{Z}_2$, where $\epsilon(v_j) = 1$ for each $v_j \in K$.

We push off the "boundary effect" one step more so that the general structure holds for $p=0$ as well!

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \underbrace{\xrightarrow{\epsilon} \mathbb{Z}_2 \xrightarrow{0}}_{} 0$$

new, as compared to the original chain complex

Notice that $\epsilon \circ \partial_1 = 0$, similar to $\partial_p \partial_{p+1} = 0 \; \forall p$, as each edges has two vertices, and both vertices have value 1 under $\epsilon$ and $1 +_2 1 = 0$.

Under $\epsilon$, a 0-cycle must have an even # vertices. As the 1-boundary of an edge (1-simplex) will have two vertices, and each of them has a value of 1 under $\epsilon$, giving $1 +_2 1 = 0$.

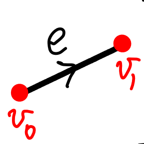
The new chain complex with homomorphisms $\partial_p$ along with $\epsilon$ gives rise to **reduced homology groups** $\tilde{H}_p(K)$. We also get the **reduced Betti numbers**, $\tilde{\beta}_p$, as the corresponding ranks: $\tilde{\beta}_p = \text{rank } \tilde{H}_p$.

Assuming $K \neq \emptyset$, $\tilde{\beta}_p = \beta_p$ for $p \geq 1$, and $\tilde{\beta}_0 = \beta_0 - 1$.

A similar augmentation map is defined for homology over $\mathbb{Z}$. $\epsilon(v_j) = 1$ for vertex $v_j$.

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0} 0 \rightarrow \cdots$$

$\epsilon \partial_1 = 0$ here as well.

$\epsilon\Big(\partial_1\Big[$  $\Big]\Big) = \epsilon(v_1 - v_0) = 1 - 1 = 0.$

# Relative Homology

One way to extend the concepts of homology from finite simplicial complexes, to pairs of spaces.

Let $K$ be a simplicial complex and $K_0 \subseteq K$ be a subcomplex. We can talk about relative chain groups as

$$C_p(K, K_0) = C_p(K) / C_p(K_0),$$

i.e., as the quotient of the chain groups in $K$ and $K_0$.

IDEA: Divide $C_p(K)$ into cosets of p-chains that differ in p-simplices in $K_0$, but not over $K - K_0$.

We can consider $\partial_p$ from $C_p(K, K_0)$ to $C_{p-1}(K, K_0)$ as the homomorphism induced by $\partial_p$ originally defined on $K$.

$$\partial_p : C_p(K, K_0) \longrightarrow C_{p-1}(K, K_0).$$

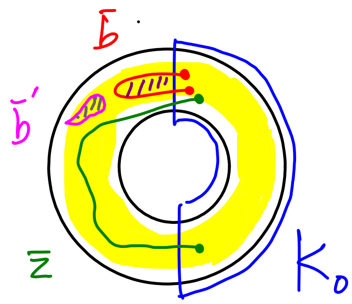
These boundary maps also satisfy $\partial_p \circ \partial_{p+1} = 0$. Hence we can define cycles, boundaries, and homology groups in the relative setting.

$$Z_p(K, K_0) = \ker \partial_p, \quad B_p(K, K_0) = \operatorname{im} \partial_{p+1}, \text{ and}$$

$$H_p(K, K_0) = Z_p(K, K_0) / B_p(K, K_0).$$

<u>Examples</u> Let $K$ be an annulus, and $K_0$ be half of it.

K




$\bar{z}$ is a 1-cycle in $K/K_0$, i.e., $\bar{z} \in Z_1(K,K_0)$.

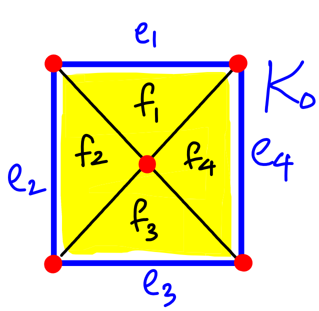
So is $\bar{b}$. But $\bar{b}$ is also a 1-boundary in $K/K_0$, i.e., $\bar{b} \in B_1(K,K_0)$.

Similarly, $\bar{b}' \in B_1(K,K_0)$, but is also in $B_1(K)$.

Intuitively, we consider everything in $K_0$ as "trivial", or as "reduced to empty". For instance, $\partial\bar{z}$ above is the set of its two end points, which are both in $K_0$. Hence, $\partial\bar{z}$ is considered empty in $K/K_0$, and hence $\bar{z}$ is a relative cycle.

Here is another example. Let $K_0$ be the four boundary edges of $K$, which is made of the four triangles.

K




$K_0 = \{e_1, e_2, e_3, e_4$, and faces thereof$\}$.

$\bar{c} = \sum_{1}^{4} a_j f_j$ is a general 2-chain of $K$

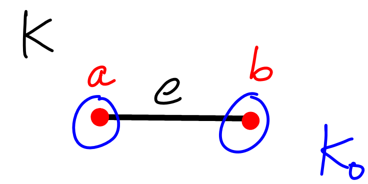
When $a_j = a \in \mathbb{Z}\ \forall j$, $\partial\bar{c} \in K_0$ (when $a=1$, we get all of $K_0$ as $\partial\bar{c}$). In this case, $\partial\bar{c}$ is not empty in $K$.

Hence, $\bar{c}$ is not a 2-cycle in $K$.

But $\bar{c} \in Z_2(K,K_0)$ as $\partial\bar{c}$ is contained in $K_0$.

Intuitively, we "shrink" all of the boundary of $\bar{c}$ to empty, thus "creating" a 2-sphere out of it!

Notice that $K_0$ need not be connected. Consider the example where $K$ consists of the single edge $e = \overline{ab}$ (and the vertices $a$ and $b$). Let $K_0 = \{a, b\}$, the two vertices. Here, $e$ is a relative 1-cycle in $(K, K_0)$.




We can compute ranks of, and bases for, relative homology groups using the same concept of SNF of $[\partial_p]$. We start off by defining the relative boundary matrix for $(K, K_0)$ as the submatrix of $[\partial_p]$ obtained by removing the rows and columns corresponding to simplices in $K_0$. The rest of the procedure is identical to the one used in the absolute, i.e., default, case.

We will use the idea of relative homology in defining some of the concepts in the upcoming lectures.