MATH 529 – Lecture 19 (03/19/2024)

Today: * persistence
* incremental algorithm for betti numbers

Persistent Homology

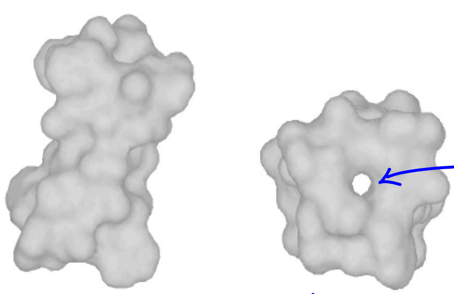
We will consider a filtered complex K $\phi = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = K$,
$|K| = \mathbb{X}$, some space.

Let $\beta_k^\ell = \text{rank}(H_k^\ell)$ where $H_k^\ell = Z_k^\ell / B_k^\ell$ with $Z_k^\ell = Z_k(K^\ell)$, etc.

We could track β_k^ℓ for fixed k as ℓ varies, getting more information than obtained from any single ℓ.

<u>Example</u> (from Edelsbrunner, Letscher, Zomorodian, 2002).

Gramicidin A, a protein with an open tunnel running through it.

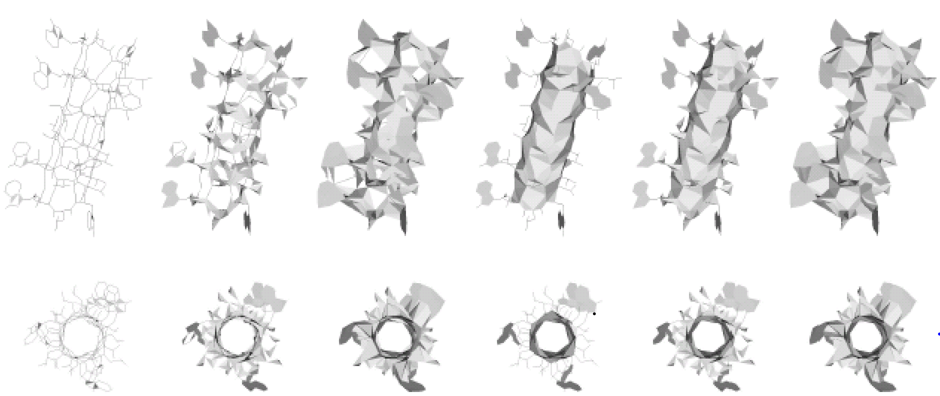


tunnel going through

side view top view

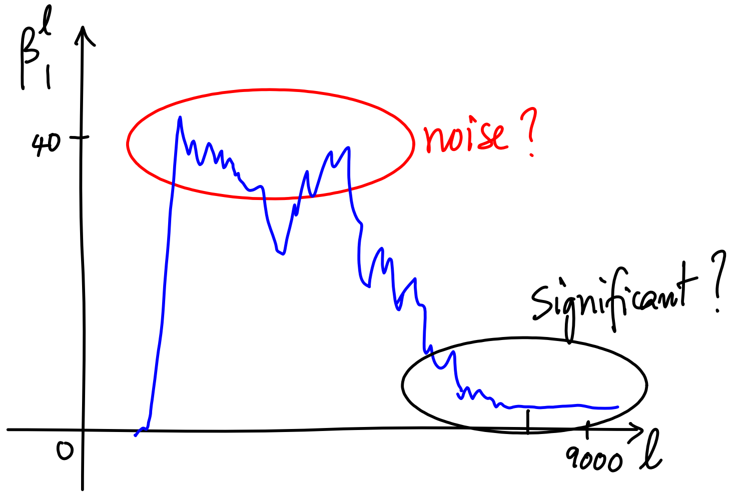
Shown below are certain subcomplexes from the <u>α-complex filtration</u> of gramicidin.

→ with atoms as points, having different starting radii.



→ side views

→ top views



The appears to be lots of holes showing up early on, which all get "closed" later on. We need to separate topological noise from features.

A "significant" feature should have a long "lifetime" in the filtration. We look for non-bounding cycles that do **not** turn into boundaries "in the near future," i.e., for the next p complexes in the filtration. Thus, we look for cycles in K^ℓ that stay non-bounding till $K^{\ell+p}$.

Def The p-persistent k^{th} homology group of K^ℓ is

$$H_k^{\ell,p} = Z_k^\ell / (B_k^{\ell+p} \cap Z_k^\ell).$$

And $\beta_k^{\ell,p} = \text{rank}\,(H_k^{\ell,p})$ is the p-persistent k^{th} betti number of K^ℓ.

Let the non-bounding k-cycle $\bar{z}$ be created at time i (for index $\ell, \ell+p$, etc.) with the arrival of simplex σ^i. So $[\bar{z}]$, the homology class represented by $\bar{z}$ is in H_k^i. Let σ^j arrive at time j, and turn $\bar{z}' \in [\bar{z}]$ into a boundary. The "k-dimensional hole" captured by $[\bar{z}]$ is "closed" by σ^j.

So $\bar{z}' \in B_k^j$, and $[\bar{z}]$ is now merged with some older class of cycles, i.e., it no longer exists independently. But it did exist independently for all times $i \le t < j$, i.e., for $j-i-1$ steps.

The **persistence** of $[\bar{z}]$ (or of $\bar{z}$ itself) is $j-i-1$, and $[i,j)$ is its **life-time** in the filtration.

σ^i is the **creator** of $[\bar{z}]$, and σ^j is the **destroyer**.

A simplex that is a creator is a positive simplex, and one that destroys is a negative simplex.

If a class has no destroyer, its persistence is $+\infty$.

<u>Time-based persistence</u>

$$H_k^{\lambda,\mu} = Z_k^{\lambda} / \left(B_k^{\lambda+\mu} \cap Z_k^{\lambda}\right) \quad \text{for times } \lambda, \mu \geq 0.$$

We could have the filtration growing with time (instead of adding one simplex in each step). By default, though, we will consider index-based persistence. The two notions are equivalent in some sense (under appropriate assumptions).

Incremental Algorithm for Betti numbers (Delfinado & Edelsbrunner, 1995)

The algorithm assumes K is in $\mathbb{R}^3$ and is torsion-free. Equivalently, K is a subcomplex of (a triangulation) of $\mathbb{S}^3 \simeq \mathbb{R}^3 \cup \{\infty\}$.

We add one simplex at each step l to grow K. Equivalently, in the filtration of K, $K^j = K^{j-1} \cup \{\sigma^j\}$ $\forall j$.

We want to classify k-simplices for $k=0,1,2,3$ as creators or destroyers. Later on we will talk about pairing a destroyer with a creator, and use these pairings to compute persistence.

To simplify notation for this discussion, we write $L = K \cup \{\sigma\}$, instead of $K^j = K^{j-1} \cup \{\sigma^j\}$. We consider four cases, corresponding to σ being a vertex, edge, triangle, or a tetrahedron.

In fact, this classification is valid in general when we add a simplex σ to a simplicial complex K (i.e., not necessarily in a filtration framework).

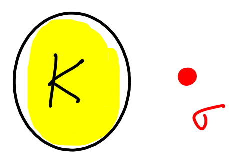
Recall that $K^{j-1} \subseteq K^j$ in the filtration. Hence when we add a simplex σ in step j, we are guaranteed that all proper faces of σ are already present in the previous complex.

For instance, when we add $\sigma = \Delta abc$, the 0-simplices a, b, c, and the 1-simplices ab, ac, and bc are already present.

A simple way to create such a filtration is to order all simplices by their dimensions, with lower dimensional simplices appearing before higher dimensional ones (we can break ties arbitrarily). Standard filtrations such as alpha or Vietoris-Rips complexes also provide such orderings once we use dimension to break ties among simplices (so, if a triangle and its faces are present at a given radius, the edges come before the face).

Case 1. σ is a 0-simplex

We get $\beta_0(L) = \beta_0(K) + 1$.

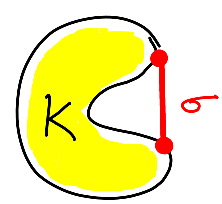


σ here adds a new class of 0-cycles. Other homology classes are not changed. Thus, 0-simplices are always creators, i.e., they are always positive.

Case 2 σ is a 1-simplex

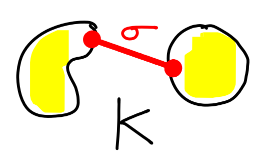
If σ belongs to a 1-cycle in L,

$$\beta_1(L) = \beta_1(K) + 1,$$



else

$$\beta_0(L) = \beta_0(K) - 1.$$



Other β_k's are not affected.

So, 1-simplices can be creators or destroyers.

Case 3 σ is a 2-simplex

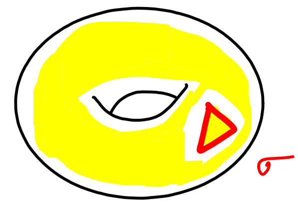
If σ belongs to a 2-cycle in L

$$\beta_2(L) = \beta_2(K) + 1;$$

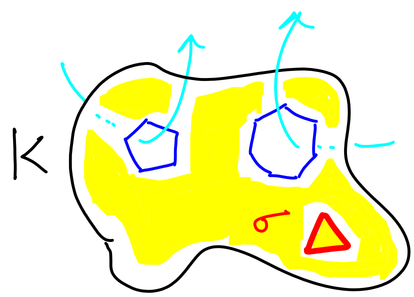
else

$$\beta_1(L) = \beta_1(K) - 1.$$

(other β_k's remain unchanged)



all other triangles in the surface of the torus are already present, and σ just closes the final hole, thus trapping the 3D space inside.



σ closes one of the holes.

Thus, 2-simplices can be creators or destroyers, just like 1-simplices.

Case 4 σ is a 3-simplex.

$$\beta_2(L) = \beta_2(K) - 1.$$

The tetrahedron coming in closes the void formed by its four faces.

So, 3-simplices are always destroyers (or, are negative) in our setting.

Since we assumed that K is a subcomplex of a triangulation of S^3 in $\mathbb{R}^3$, we do not get any 3-cycles. So, $\beta_3(K) = 0$.

IDEA: To understand the above statement about $\beta_3=0$, we will go down in dimension by 1. Think about K being a subcomplex of $\mathbb{S}^2$ and consider triangles. The only case when a triangle is positive is when it comes in to complete the surface of $\mathbb{S}^2$. But, if we further require K to be embedded in $\mathbb{R}^2$, then we cannot have a 2-cycle and all triangles are negative $(\beta_2(K)=0)$. The case of tetrahedra in K that is a subcomplex of a triangulation of $\mathbb{S}^3$, sitting in $\mathbb{R}^3$, is similar (we get $\beta_3(K)=0$).

The idea of classifying simplices as positive and negative can be extended to arbitrary dimensions, though.

Algorithm for $(\beta_0,\beta_1,\beta_2)$ when $K \subset$ triangulation of $\mathbb{S}^3$

integer^3 BETTI

$\beta_0=\beta_1=\beta_2=0$;

for $j=1$ to m do

filtration is $\underset{\emptyset}{\overset{\shortparallel}{K^0}} \subseteq \cdots \subseteq K^m = K$ (m simplices in K)

$k = \dim \sigma^j$;

$\boxed{\text{if } \sigma^j \text{ belongs to a } k\text{-cycle in } K^j \text{ then}}$ non-trivial in general!

β_k++; $(\beta_k \leftarrow \beta_k+1)$

else

β_{k-1}−−; $(\beta_{k-1} \leftarrow \beta_{k-1}-1)$

endif

endfor

return $(\beta_0,\beta_1,\beta_2)$;

We can understand the logic behind this algorithm using the formulae for Betti numbers. Recall,

$$\beta_k = z_k - b_k \quad \text{— (1)}$$

$$\beta_{k-1} = z_{k-1} - b_{k-1} \quad \text{— (2)}$$

$$s_k = z_k + b_{k-1} \quad \text{— (3)}$$

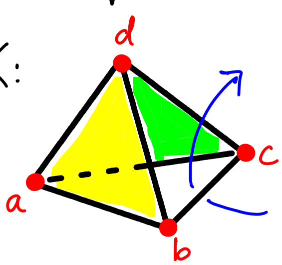
When we add a k-simplex, s_k goes up by 1, which we indicate by $s_k\uparrow$.

If $s_k\uparrow$, then (3) $\Rightarrow$ either $z_k\uparrow$ or $b_{k-1}\uparrow$.

If $z_k\uparrow$, then (1) $\Rightarrow$ $\beta_k\uparrow$; and

if $b_{k-1}\uparrow$, then (2) $\Rightarrow$ $\beta_{k-1}\downarrow$.

Example

K: 

filtration: a, b, c, d, ab, ac, ad, bc, bd, cd, abd, acd $(m=12)$

There is one connected component, and one hole.