MATH 529 – Lecture 25 (04/09/2024)

Today: * optimality in homology
* homology over $\mathbb{Z}$

Optimality in Homology

We consider homology groups over $\mathbb{Z}$ to define optimization problems. While $\mathbb{Z}_2$ might be the simpler ring to consider, it turns out $\mathbb{Z}$ opens up other possible approaches not possible with $\mathbb{Z}_2$ in this context.

Homology over $\mathbb{Z}$

$\mathbb{Z}$ has both $+z$ and $-z$ for any integer z. $\mathbb{Z}_2$ has only 1 (no $\pm$ signs)

We can use orientation of simplices/chains to capture the $\pm$ signs.

Recall induced orientation

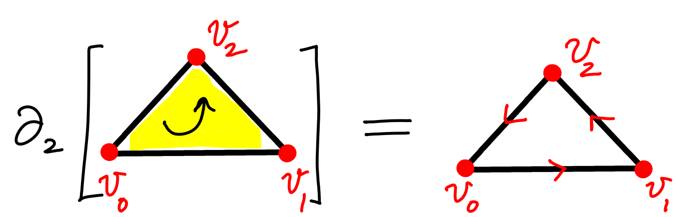
v_i is omitted

Let $\sigma = [v_0 v_1 \cdots v_p]$ be a p-simplex and let $\tau = \text{conv}\{v_0, \ldots \hat{v}_i, \ldots v_p\}$ be a $(p-1)$-face. The orientation induced on τ by σ is the same as $[v_0 \ldots \hat{v}_i \ldots v_p]$ if i is even.

The boundary of a simplex is defined as follows.

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \, [v_0 \ldots \hat{v}_i \ldots v_p].$$

For example,

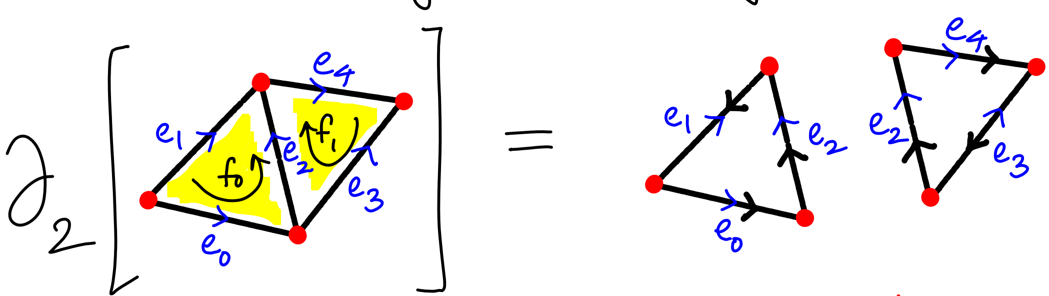


$[v_1 v_2] - [v_0 v_2] + [v_0 v_1]$

For a p-chain $\bar{c} = \sum_{j=1}^{n} a_j \sigma_j$, where $a_j \in \mathbb{Z}$ and σ_j's are p-simplices,

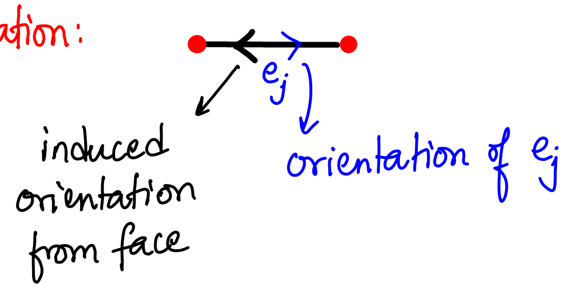
$$\partial_p \bar{c} = \partial\left(\sum_{j=1}^{n} a_j \sigma_j\right) = \sum_{j=1}^{n} a_j (\partial \sigma_j).$$

Choices of orientations for p-simplices become critical in boundary computations when working over $\mathbb{Z}$. Consider the 2-chain consisting of two triangles shown below.



$$\bar{c}_1 = \begin{array}{c} f_0 \\ f_1 \end{array}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

notation:



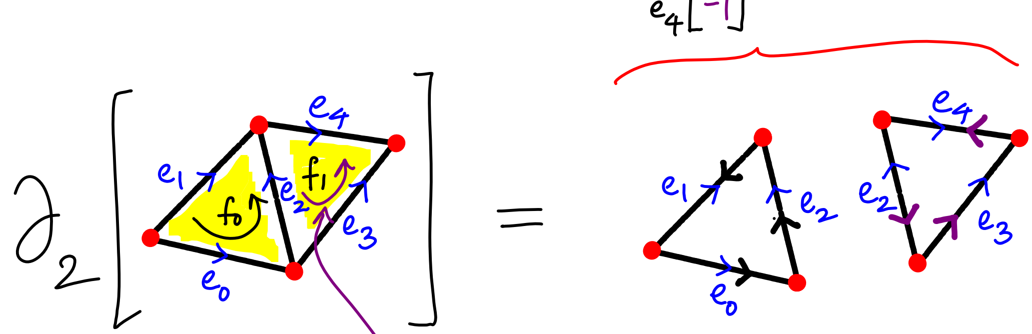
$$\partial_2 \bar{c}_1 = \bar{b}_1 = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Notice that f_0 and f_1 are not consistently oriented. Induced orientations on e_2 are same, and hence e_2 has weight 2 in $\partial_2 \bar{c}_1$.

When working over $\mathbb{Z}_2$, e_2 will be counted twice, and hence will cancel when taking $\partial_2 \bar{c}_1$.

But, consider $\partial \bar{c}_2$ where $\bar{c}_2 = \begin{array}{c} f_0 \\ f_1 \end{array}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We get $\partial \bar{c}_2 = \bar{b}_2 = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$



$$\bar{c}_2 = \begin{array}{c} f_0 \\ f_1 \end{array}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

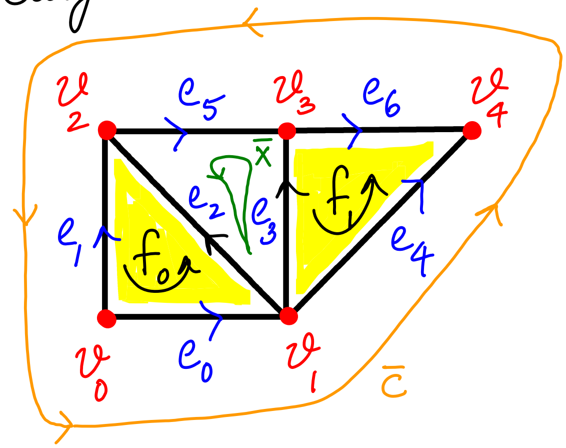
f_0 and f_1 are consistently oriented in $\bar{c}_2$, and hence e_2 cancels when taking boundary. e_2 will cancel as long as the weights for f_0 and f_1 are same in absolute value, and f_0 and f_1 are consistently oriented.

The definitions of C_p, Z_p, B_p, and H_p are similar to what we have seen before (when we were working over $\mathbb{Z}_2$). So, $\partial_p : C_p \to C_{p-1}$ are the boundary homomorphisms with addition over $\mathbb{Z}$ now, and $Z_p = \ker \partial_p$, $B_p = \operatorname{im} \partial_{p+1}$. Similarly, $H_p = Z_p / B_p$.

Let us consider our popular example now. We orient the triangles CCW, and edges lexicographically.

We now consider 1-cycles in K. In particular, we consider representative cycles around the hole. Each cycle is also oriented here.

K:



Consider the two cycles $\bar{c}$ and $\bar{x}$ shown here which both go CCW around the hole in K. As vectors, we can write them as follows.

$$\bar{c} = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad \bar{x} = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

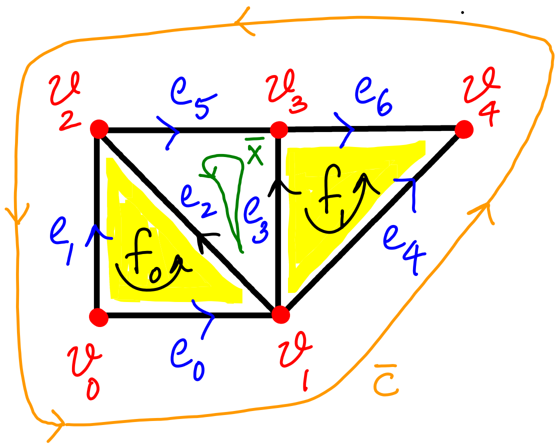
e_1 is directed opposite to the orientation of $\bar{c}$

$\bar{c}$ and $\bar{x}$ are both cycles around the same hole, and hence are homologous. Similar to how we wrote such homology relations over $\mathbb{Z}_2$ using the boundary matrix, we form the boundary matrix over $\mathbb{Z}$ here. Now, the boundary matrices have entries in $\{-1,0,1\}$.

Compare with $[\partial_p]$ over $\mathbb{Z}_2$, which had entries in $\{0,1\}$.

$$[\partial_2] = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cc} f_0 & f_1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

K:



In general, $[\partial_p]$ is an $m \times n$ matrix when K has m $(p-1)$-simplices τ_i and n p-simplices σ_j. The (i,j)-entry is nonzero if $\tau_i \leq \sigma_j$, and 0 otherwise. This nonzero entry is $+1$ if orientations of τ_i and σ_j agree, and -1 if they are opposite.

Homology between $\bar{c}$ and $\bar{x}$ can be written in terms of $[\partial_2]$, as follows.

$$\bar{x}: \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \bar{c}: \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \underbrace{\begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cc} f_0 & f_1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{array} \begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{[\partial_2]\bar{y}} \quad \text{where } \bar{y} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

So, $\bar{x} \sim \bar{c}$ here.

Add boundaries of two triangles f_0, f_1, after reversing their orientations, to $\bar{c}$ to get $\bar{x}$.

Note that $\bar{c}$ has 5 edges and $\bar{x}$ has 3. Even if we were to include (Euclidean) lengths of the edges, and define

$$\|\bar{c}\| = \sum_{i=0}^{6} w_i |c_i| \quad \text{where} \quad w_i = 1 \text{ for horizontal/vertical edges } \{0,1,3,5,6\}$$
$$= \sqrt{2} \text{ for diagonal edges } \{2,4\}$$

We see that $\|\bar{x}\| = 2+\sqrt{2}$ and $\|\bar{c}\| = 4+\sqrt{2}$.

More generally, we could take w_i as the length of an edge, or the area of a triangle. Irrespective of the direction of traversal of an edge, we want to count its length — hence the absolute value $|x_i|$. We can seek a minimal weight chain $\bar{x}$ that is homologous to $\bar{c}$.

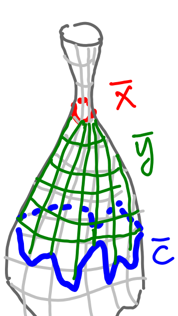
could generalize to chain when seeking $\bar{x} \sim \bar{c}$ for a chain $\bar{c}$.

Optimal Homologous Cycle Problem (OHCP)

Given $\bar{c}$ in $[\bar{c}] \in H_p(K)$, find $\bar{x} \in [\bar{c}]$ with $\|\bar{x}\| = \sum_{i=1}^{m} w_i |x_i|$, where $w_i \geq 0$ is the weight for σ_i the i^{th} p-simplex, is smallest.

We can prove an optimal homologous cycle always exists assuming K (simplicial complex) is finite, and $w_i \geq 0 \ \forall i$.

We can cast OHCP as the following optimization problem:



$$\text{minimize} \quad \sum_{i=1}^{m} w_i |x_i| + \lambda \left(\sum_{j=1}^{n} v_j |y_j| \right)$$
$$\text{subject to} \quad \bar{x} = \bar{c} + [\partial_{p+1}] \bar{y}$$
$$\bar{x} \in \mathbb{Z}^m, \ \bar{y} = \mathbb{Z}^n$$

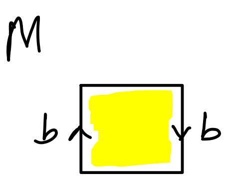
generalization where we also minimize the area of triangles used to define $\bar{x} \sim \bar{c}$.

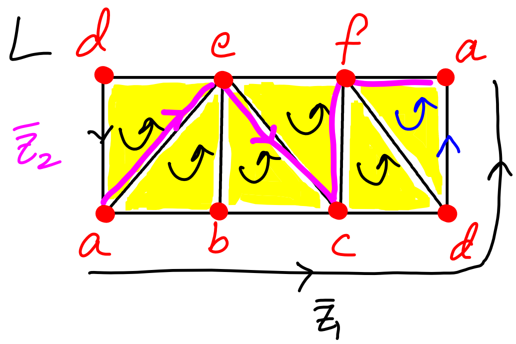
$\lambda \geq 0$ is a scale parameter that controls the relative importance of the two terms in the objective function.

Working over $\mathbb{Z}$ instead of $\mathbb{Z}_2$ gives us more "freedom", and actually makes the OHCP easier to solve under certain settings!

Homology over $\mathbb{Z}$ for some familiar objects

1. Möbius strip





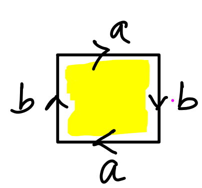
$\bar{z}_1 = a\text{-}b\text{-}c\text{-}d\text{-}a$

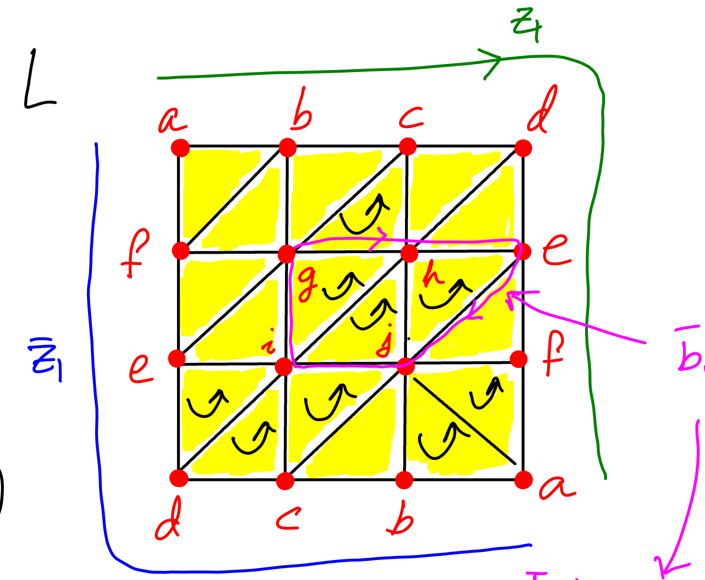
$\bar{z}_2 = a\text{-}e\text{-}c\text{-}f\text{-}a$

$\bar{z}_1 - \bar{z}_2 = \partial_2 \bar{d}$, where $\bar{d}$ is the 2-chain of triangles between $\bar{z}_1$ and $\bar{z}_2$.

We could go around $\bar{z}_1$ any integer # times. Hence $H_1 \simeq \mathbb{Z}$, $\{\bar{z}_1\}$ is a basis, and $\beta_1(L;\mathbb{Z}) = \text{rank}(H_1;\mathbb{Z}) = 1$ here. $H_0(L;\mathbb{Z}) \simeq \mathbb{Z}$ as well (could use any vertex as a basis. Here, $\{\beta_0, \beta_1\}$ over $\mathbb{Z}$ coincide with those over $\mathbb{Z}_2$.

2. Projective Plane





$\bar{b}_1$ is a 1-boundary, bounding the 3 triangles inside.

$H_0(L;\mathbb{Z})$ has same rank as $H_0(L;\mathbb{Z}_2)$ ($\beta_0 = 1$ over both $\mathbb{Z}$ and $\mathbb{Z}_2$)

There is 1 connected component, so $H_0(K;\mathbb{Z})$ could be expected to be similar to $H_0(K;\mathbb{Z}_2)$.

Let's consider H_1.

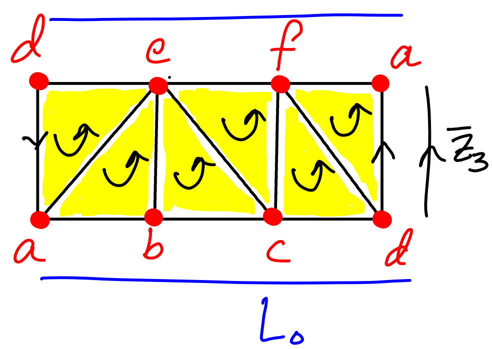
We can start by assigning CCW orientations for all triangles. Then the induced orientations for all edges "in the middle" are indeed opposite.

Consider the cycle $\bar{z}_1 = abcdefa$. It turns out that $2\bar{z}_1$ is a boundary — it bounds the 2-chain that has all triangles with coefficient 1

So, we could repeat $\bar{z}_1$ an odd # times to get another cycle in H_1. But an even # times gives a boundary. So, here $H_1 \cong \mathbb{Z}_2$.

Back to Example 1: Möbius strip

L



Let $L_0 \subset L$ consist of the edges $\overline{ab}, \overline{bc}, \overline{cd}, \overline{de}, \overline{ef}, \overline{fa}$, and the vertices. Let $\bar{z}_3 = \overline{ad}$.

The induced orientations on $\overline{ad}$ from $\triangle ade$ and $\triangle adf$ do not cancel.

What is $H_1(L, L_0; \mathbb{Z})$?

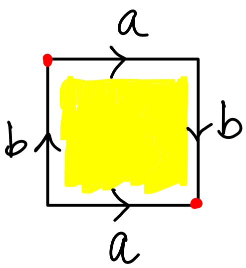
Note that all edges going "across" ($\overline{ad}, \overline{ae}, \overline{be}, \overline{ce}$, etc.) are relative 1-cycles. Also, if $\bar{d}$ is the 2-chain with coefficients 1 for each triangle, then $\partial_2 \bar{d} = 2\bar{z}_3$. Thus, $\bar{z}_3$ is not a relative 1-boundary, but $2\bar{z}_3$ is.

$$\Rightarrow \quad H_1((L, L_0); \mathbb{Z}) \cong \mathbb{Z}_2.$$

We will see later that Möbius strips are a canonical way in which homology groups that are not $\cong \mathbb{Z}$ show up in surfaces, and in 2-complexes more generally.

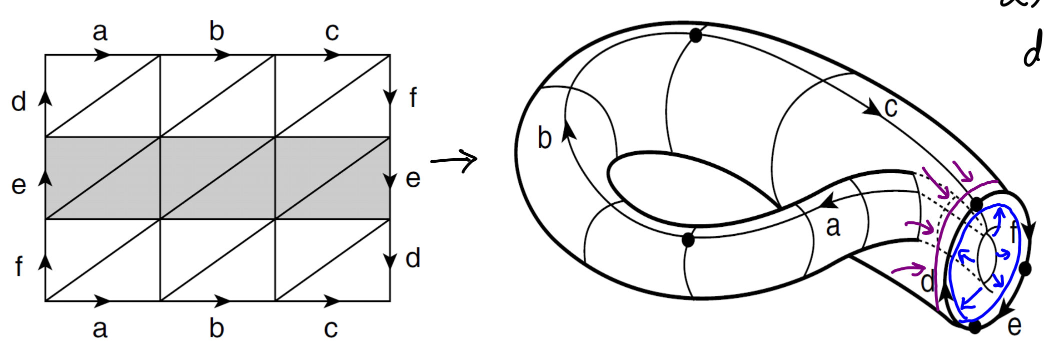
In fact, the "canonical" way in which a 2-manifold (w/ or w/o boundary) is non-orientable is by having a Möbius strip as a subcomplex.

3. Klein Bottle ($\mathbb{K}^2$)

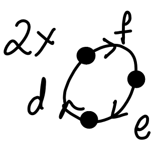


Notice that gluing along b produces an "intermediate" Möbius strip (before we also glue along a). Going multiple times around a shows the same behavior — a essentially behaves like the "skeleton" of this Möbius strip, thus capturing the main "hole" of the strip.

But, going around b twice after gluing along a gives a boundary — the boundary of the entire piece of paper. See the alternative triangulation and gluing below.



(images from P.J. Giblin – "Graphs, Surfaces, & Homology")

2x 

forms the boundary of the entire surface, when taken twice — once for the part "outside" and the second for the part "coming through".

Notice that going just once around gives a cycle that is not a boundary.

Hence $H_1(\mathbb{K}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$

"free part" of H_1 ($\mathbb{Z}$)

"torsion part" of H_1 ($\mathbb{Z}_2$)

$\mathbb{Z}_2$ also written as $\mathbb{Z}/2$