MATH 529 – Lecture 28 (04/18/2024)

Today:
* OHCP and TU
* torsion
* other optimal homology problems

Recall: Theorem 3 $[\partial_2(K)]$ is TU iff K has no Möbius strips.

$$MCM(n) = \begin{bmatrix} 1 & & & \alpha \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \quad \alpha = (-1)^{n+1}$$

A similar (sub)matrix could be defined for a cylinder subcomplex, called cylinder cycle matrix (CCM).

$$CCM(n) = \begin{bmatrix} 1 & & & \alpha \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix} \qquad \alpha = (-1)^n.$$

$$\det(CCM(n)) = 0.$$

Notice that the corresponding cylinder subcomplex is orientable, while the Möbius subcomplex is not.

Recall examples B, B' from Lecture 27 when we introduced TU:
$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & & -1 \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}.$$

Note that $B = MCM(3)$, $B' = CCM(3)$

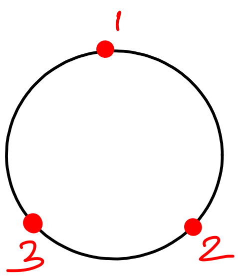
Thus, Möbius strips are canonical shapes that correspond to non-total unimodularity. How about the result in general?

Theorem 4 $[\partial_{p+1}(K)]$ is TU iff K has no relative torsion in $(p, p+1)$-dimensions.

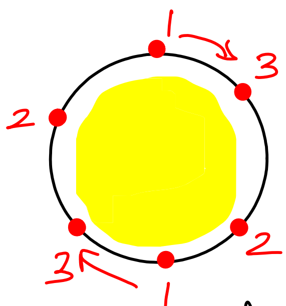
↳ torsion in relative homology groups

Torsion

Here is how Poincaré visualized torsion, a concept he originally introduced in late 1890s. Consider gluing the boundary of a disc to a circle. If we do it in the simple manner, we get a hat. But if we do the gluing such that the boundary of the disc goes twice around the circle, we get the 2-fold dunce hat, which is the projective plane $\mathbb{RP}^2$. If we go k-times around, we get the k-fold dunce hat. And k-fold dunce hats have torsion in its H_1 (first homology group). → $H_1 \cong \mathbb{Z}_k$ here.



circle

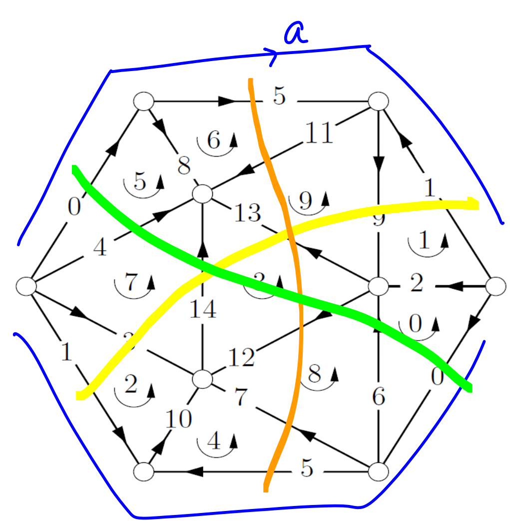


disc

2-fold Dunce hat $\approx \mathbb{RP}^2$

k-fold dunce hat in general, for $k \geq 2$.

Here is a triangulation of the 2-fold dunce hat ($\mathbb{RP}^2$). Notice the boundary edges going 0-5-(-1) twice around. Indeed, we could spot at least 3 Möbius strips, as indicated here. We could spot other Möbius strips that wander left and right more...



Theorem 4 gives an algebraic topological characterization of TU. We saw the algebraic definition (all subdeterminants are $0, \pm 1$) as well as a geometric characterization (all vertices of LP polytope have integer coordinates) previously.

How useful is this characterization? Could we check this condition efficiently?

YES. Can check if $[\partial_{p+1}]$ is TU or not in polynomial time.

Seymour's matroid decomposition (1980) $\left(O(n^3)\right)$.

Truemper - practical algo (2012) $\left(O(n^5)\right)$.

→ Check out https://discopt.github.io/cmr/ .

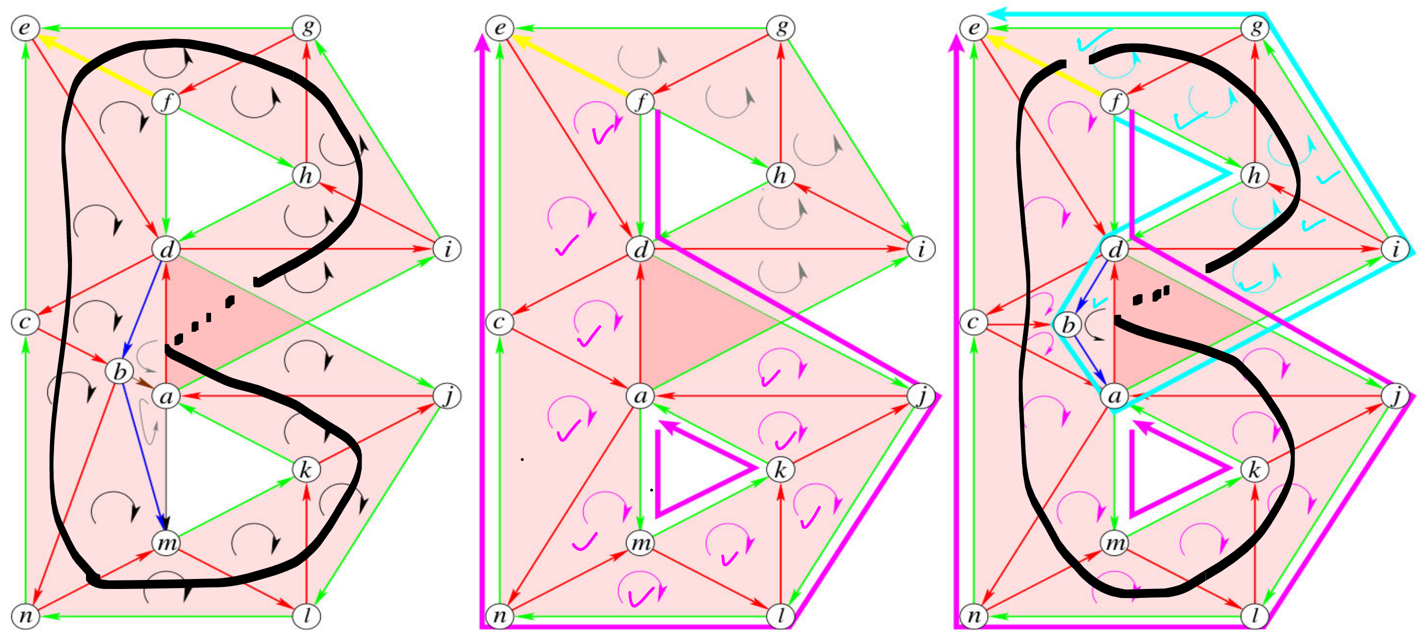
Here is a summary of the special cases of OHCP that are easy.

1. $[\partial_{p+1}]$ is TU if K is an orientable $(p+1)$-manifold.
2. $[\partial_2]$ is TU if K has "no Möbius strip".
3. $[\partial_d]$ is TU for K in $\mathbb{R}^d$. → tetrahedra-triangle case in $\mathbb{R}^3$ is easy!

Result 3 is similar to Result 1: We cannot realize/embed a Möbius strip in $\mathbb{R}^2$ for instance. But the proof presented used different techniques.

More recently, people have studied optimal homology problems over a filtration (rather than on a single simplicial complex).

If K is not TU, is all hope lost? No!
Non-Total Unimodularity Neutralized (NTUN) complexes!



not TU TU not TU

There are Möbius strips (as subcomplexes) in both the first and third complexes, while the middle (second) complex has a TU $[\partial_2]$ matrix. But we are guaranteed to find an integer solution for every OHCP instance of K_3, as it is NTU-neutralized.

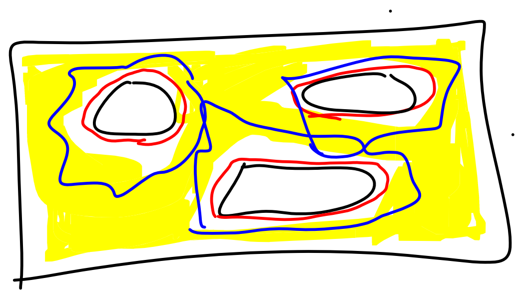
See the paper for details, at http://arxiv.org/abs/1304.4985

Being NTU neutralized is a property of K, and not of $|K|$, as the above example illustrates.

There are several interesting open problems in this context. For instance, **all** instances of OHCP for 1D inputs in a convex 3D domain, e.g., 2-skeleton of a solid unit cube in computations were solved by their corresponding LPs – there ought to be some deeper theory that's waiting to be unearthed!

Related Optimal homology problems

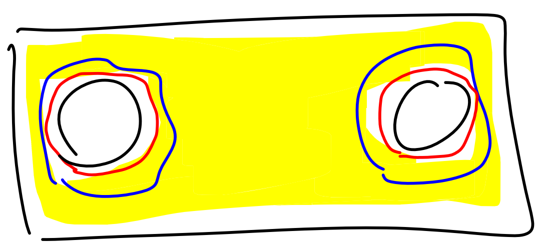
The Optimal Homology Basis Problem (OBAS)



Given a loop around each hole, find a set of tightest loops around each hole, or a minimal basis in general.

Q: Could we solve OBAS as an OHCP?

What complexes exist for which the OHCP solution with the collection of cycles in a homology basis as input is actually an optimal homology basis?



May be we could characterize the case where the holes are "far apart" from each other, and the loops around each hole are also disjoint. In this case, solving the OHCP with the collection of input cycles as the single input cycle should solve the OBAS problem.