MATH 529 — Lecture 30 (04/25/2024)

Today: * integral decompositions of currents
* median shapes

How do we reconcile continuous currents with simplicial ones?

There are results on how well the simplicial chains approximate the (continuous) currents — **simplicial deformation theorem.**

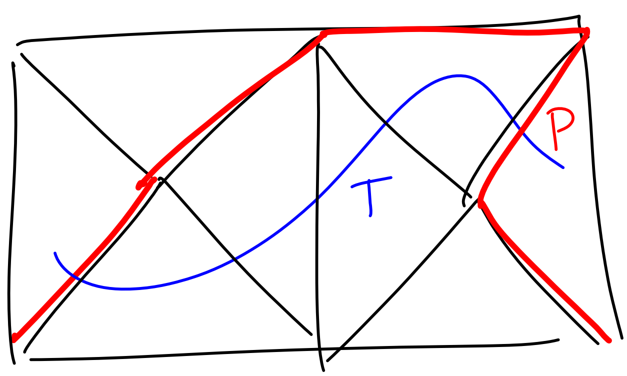
Any current T can be deformed to a simplicial chain P in a simplicial complex K such that

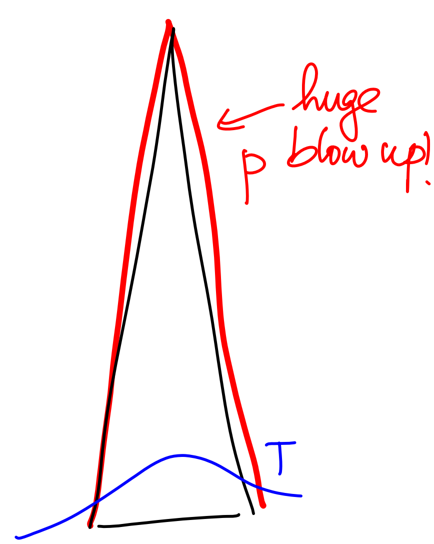
$$M(P) \le c_1 M(T) + \Delta\, c_2 M(\partial T)$$

$$M(\partial P) \le c_2 M(\partial T), \quad \text{and}$$

$$\mathbb{F}(T,P) \le \Delta\, c_1 \left[M(T) + c_3 M(\partial T) \right]$$

$c_1, c_2, c_3 > 1$ constants, and $\Delta \to 0$ as we take finer and finer simplicial complexes.





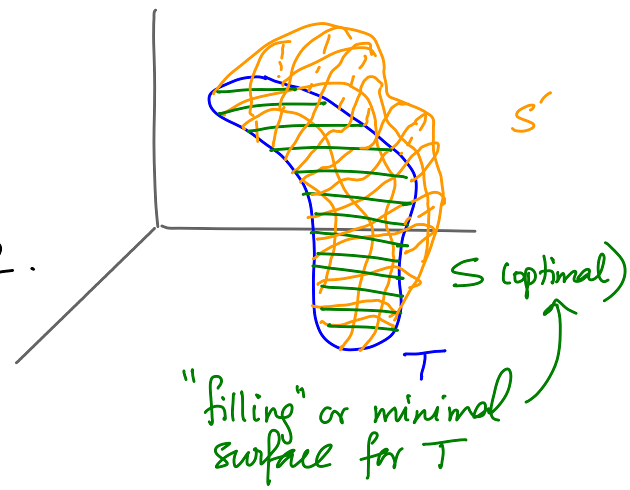
In the trivial case, T could be already simplicial, and we don't have to do any pushing at all. But in the general setting, the regularity of the complex will affect our deformation. Hence c_1, c_2, c_3 are >1, and could be large, in fact.

TU connection to Prove Fundamental Results

Apart from efficient computability of optimal homologous chains, the total unimodularity result could also be used to prove (more) fundamental results.

Here is a motivating problem:

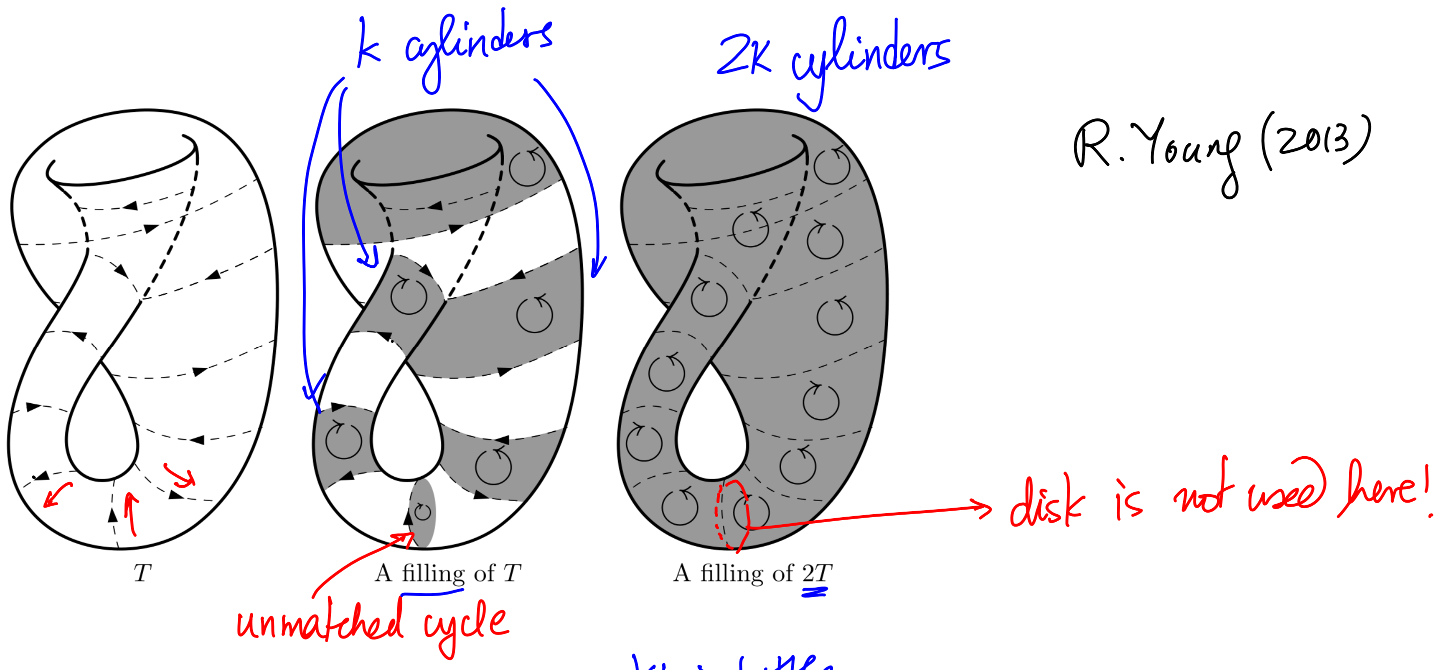
Q. When is the minimal surface for $2T$ the same as 2(minimal surface for T)?



The answer is YES in codimension 1 and 2.

But No in codimension 3!

Here is a counterexample.



R. Young (2013)

T: $2k+1$ cycles around $\mathbb{K}^2$ (Klein bottle), with adjacent cycles oriented oppositely.

Area of min surface for $2T$ < 2(area of min surface for T).

A similar question on flat norm decomposition:

$$\mathbb{F}_\lambda(T) = \min_S \{ M_d(\underbrace{T-\partial S}_{X}) + \lambda M_{d+1}(S) \}$$

Q: For integral T (with integer multiplicities), when is its flat norm decomposition (X, S) also integral?

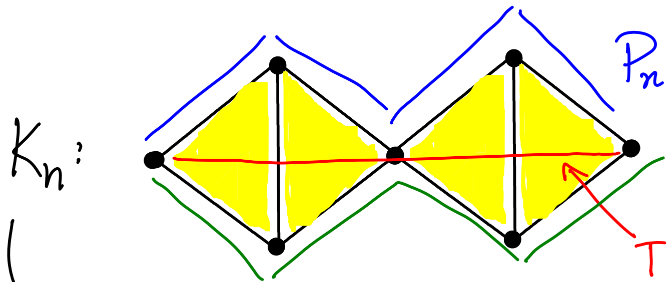
The answer is YES in the finite simplicial setting in codimension 1, as $[\partial_{d+1}]$ is TU. But the answer is not known in general.

F. Almgren considered this related problem:
"If a sequence of currents $\{2T_i\}$ converge in the integral flat topology, must the sequence $\{T_i\}$ also converge?"

B. White: "He used a lot of weapons, including his enormous (m-2)-regularity paper..."
single paper that is 1,500+ pages long!

Could we use the TU result, and then somehow take finer and finer simplicial complexes and somehow the result holds in the limit for the continuous case?

Challenges: The simplicial flat norm may not converge to the (continuous) flat norm.



2n equilateral triangles

$$\mathbb{F}_{K_n}(P_n) = \frac{2}{\sqrt{3}}\, \mathbb{F}(P_n)$$

gives T as the optimal decomposition

The simplicial flat norm is always a multiple of flat norm even as $n \to \infty$!

We could use all the triangles but end up with another chain of the same length as the "upper roof" input! While the continuous flat norm decomposition is T, the flat line.

We cannot apply simplicial deformation directly!
we want mass expansion ~~of~~ T and ∂T to $\to 0$:

$$M(P) \leq M(T) + \rho \; (*)$$
$$M(\partial P) \leq M(\partial T) + \rho \; (*)$$

and $\rho \to 0$ as we take finer and finer simplicial complexes.
Compare these statements to the bounds specified by the simplicial deformation theorem.

See the paper for details: http://arxiv.org/abs/1411.0882 .

Median Shapes

Notions of average of $x_1, \dots, x_k \in \mathbb{R}$?

mean: $\hat{x} = \underset{x}{\operatorname{argmin}} \sum_{i=1}^{k} (x - x_i)^2$

median: $\hat{x} = \underset{x}{\operatorname{argmin}} \sum_{i=1}^{k} |x - x_i|$ → less sensitive to outliers

Generalizing to vectors: geometric median of $\bar{x}_1, \dots, \bar{x}_k \in \mathbb{R}^d$:

$$\hat{\bar{x}} = \underset{\bar{x}}{\operatorname{argmin}} \sum_{i=1}^{k} \|\bar{x} - \bar{x}_i\|$$ → we get mean if minimizing $\sum_i \|\bar{x} - \bar{x}_i\|^2$

We defined the flat norm distance between currents. It's natural to use that distance to define an average shape of input shapes. We concentrate on the median shape.

Median of shapes represented as currents $T_1, \dots, T_k$?

$$\hat{T} = \underset{T}{\operatorname{argmin}} \sum_{i=1}^{k} \mathbb{F}_\lambda(T, T_i)$$ → we get the mean shape if we minimize $\sum_{i=1}^{k} \mathbb{F}_\lambda(T, T_i)^2$

We can write down the median shape computation as an optimization problem similar to the OHCP and flat norm in the finite simplicial setting. We specify the flat norm decomposition of each T_i w.r.t. T as a homology equation.

$$\bar{t} - \bar{t}_h = \bar{r}_h + [\partial_{p+1}]\bar{s}_h \quad , \quad h = 1, \dots, k.$$ → flat norm decomposition of each $\bar{t}_h$

(we'll need i, j, as other indices soon. Hence the switch to h ☺!).

Consider the simplicial complex with m and n p- and $(p+1)$-simplices, respectively. We use the standard approach to handle absolute value terms in the minimization objective function.

$$\min \sum_{h=1}^{k} \left(\sum_{i=1}^{m} w_i |r_{hi}| + \lambda \sum_{j=1}^{n} v_j |s_{hj}| \right)$$

$$\text{s.t.} \quad \bar t - \bar t_h = \bar r_h + B \bar s_h, \quad h = 1, \ldots, k$$

$$\bar t \in \mathbb{Z}^m, \ \bar r_h \in \mathbb{Z}^m, \ s_h \in \mathbb{Z}^n$$

In a modified version, we consider median shape with mass regularization — $|t_i| \rightarrow t_i^+ + t_i^-$ is used in the objective function in this case.

$$\min \sum_{h=1}^{k} \left(\sum_{i=1}^{m} w_i (r_{hi}^+ + r_{hi}^-) + \lambda \sum_{j=1}^{n} v_j (s_{hj}^+ + s_{hj}^-) \right)$$

$$\text{s.t.} \quad \bar t^+ - \bar t^- - (\bar r_h^+ - \bar r_h^-) - B(\bar s_h^+ - \bar s_h^-) = \bar t_h, \quad h = 1, \ldots, k$$

$$\bar t^+, \bar t^-, \bar r_h^+, \bar r_h^- \geq \bar 0_m, \quad \bar s_h^+, \bar s_h^- \geq \bar 0_n$$

$$\bar t^+, \bar t^-, \bar r_h^+, \bar r_h^- \in \mathbb{Z}^m, \quad \bar s_h^+, \bar s_h^- \in \mathbb{Z}^n$$

The constraint matrix A has the following form:

$$\begin{array}{ccccc} \bar t^+ \ \bar t^- & \bar r_1^+ \ \bar r_1^- \ \bar s_1^+ \ \bar s_1^- & \bar r_2^+ \ \bar r_2^- \ \bar s_2^+ \ \bar s_2^- & \cdots & \bar r_k^+ \ \bar r_k^- \ \bar s_k^+ \ \bar s_k^- \end{array}$$

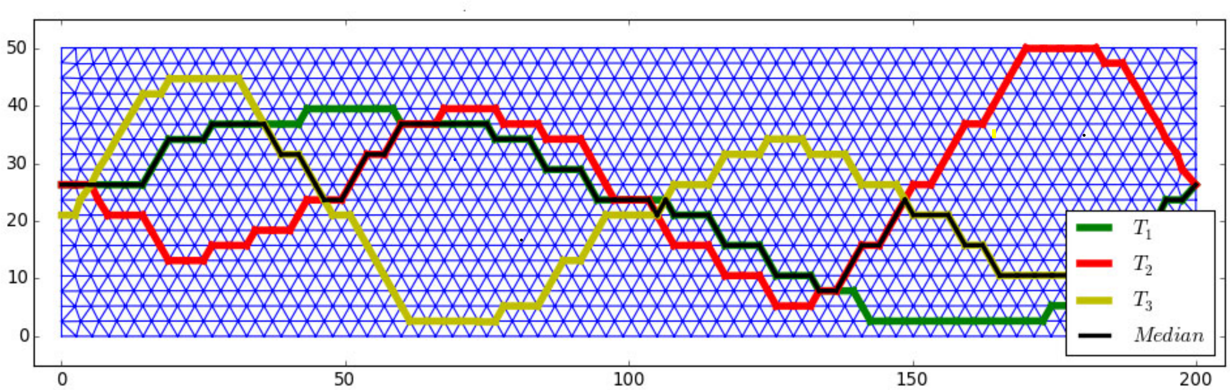
$$A = \left.\begin{bmatrix} [I \ -I] & [-I \ I \ -B \ B] & & & \\ [I \ -I] & & [-I \ I \ -B \ B] & & \\ \vdots & & & \ddots & \\ [I \ -I] & & & & [-I \ I \ -B \ B] \end{bmatrix}\right\} k \text{ blocks}$$

Because of the common $[I \ -I]$ columns $(\bar t^+, \bar t^-)$, A may not be TU even when B is! Still, every instance of this median shape LP yields integer solutions in practice!

See the paper for details: https://arxiv.org/abs/1802.04968 .
(Theoretical and computational results).

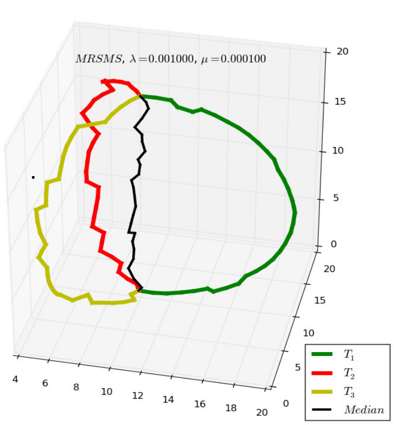
Here are some computational experiments.

1. Input is 3 (simplicial) curves in $\mathbb{R}^2$:



2. Similar to OHCP and flat norm LP producing integer solutions (for free) in codimension 1 (1-d input in $\mathbb{R}^3$), the median shape LP works as well!

The input is 3 semi-circular curves connecting the North and South poles with some noise on the surface of S^2. They are separated by $\sim 120°$ angles @ the poles. The domain is the 3-ball (solid) tetrahedralized. We get the "axle" from North to South pole as the median shape (with mass regularization).

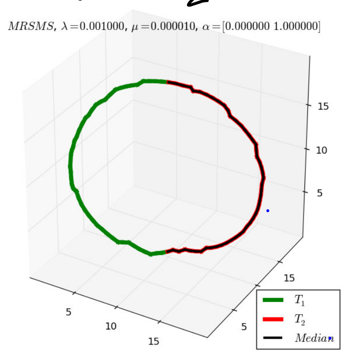


The simplicial complex (2-skeleton) is not shown for clarity reasons.

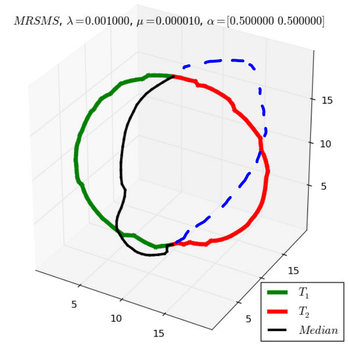
The LP framework for median shape is quite versatile, and can model several variants of the default median shape problem. One variant considers weighing the flat distances from input T_h by α_h $(0 \le \alpha_h \le 1)$ such that $\sum_{h=1}^{k} \alpha_h = 1$. This approach could be used to traverse the entire shape space of all T_h's in a convex combination fashion. We could also restrict the candidates to lie on a subspace, e.g., $\mathbb{S}^2$ instead of $\mathbb{B}^3$.

3. Two curves that form great semi-circles from North to South poles on $\mathbb{S}^2$, median shape sought on $\mathbb{S}^2$ as well.

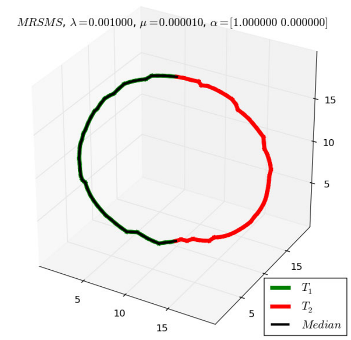
$\alpha_1 = 0 \quad \alpha_2 = 1$



$\alpha_1 = \alpha_2 = \frac{1}{2}$



$\alpha_1 = 1, \alpha_2 = 0$



there is non-uniqueness here. One could take the great semicircle going on the "other" side as the median shape as well!!