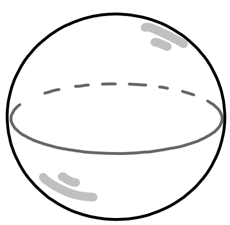
# MATH 529 – Lecture 3 (01/16/2024)

Today: * 1 more example of homeomorphism
* manifolds

## Examples of homeomorphism (continued...)

5. sphere $\not\approx$ $\mathbb{R}^2$
$\mathbb{S}^2$ (2-sphere)



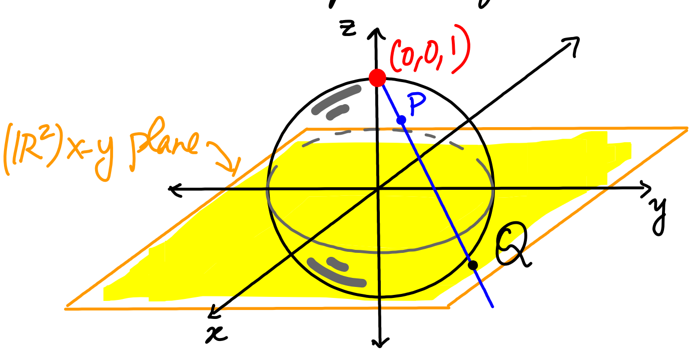
Here is an observation: the sphere encloses a 3D pocket (or void), while $\mathbb{R}^2$ does not do so.

Such enclosed voids are the 3D analogues of holes (which are 2D).

But $\mathbb{R}^2 \cup \{\infty\} \approx \mathbb{S}^2$ → By stereographic projection, which is used to map out the surface of the earth on to a (planar) map for instance.

"point at infinity"

Recall the equation for $\mathbb{S}^2$: $x^2+y^2+z^2=1$.




If you poke a hole in the sphere, you can spread it out on the 2D plane, like a pierced balloon.

The equation of the line connecting $(0,0,1)$ and $P(x,y,z)$ is given by
$\bar{x} = (0,0,1) + t\,(x-0, y-0, z-1)$, $t \in \mathbb{R}$.
This line intersects the $x$-$y$ plane at $Q$, which has $z=0$. Hence we get $t(z-1)+1=0 \Rightarrow t = \frac{1}{1-z}$.
Thus $Q$ is $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

$P(x,y,z)$ on $\mathbb{S}^2$ projected from $(0,0,1)$ → North pole to $\mathbb{R}^2$ is

$$Q\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

This formula is valid for all points on $\mathbb{S}^2$, except the north pole $(0,0,1)$.

The lower hemisphere of $\mathbb{S}^2$ gets mapped to $D$ (unit disc), and the upper hemisphere gets mapped to rest of $\mathbb{R}^2$.

According to a topologist, "a sphere is nothing but the plane with a point added at infinity"!

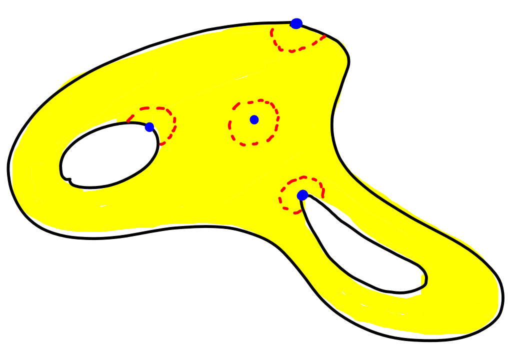
Note that every point on $\mathbb{S}^2$ has a neighborhood that looks like $\mathbb{R}^2$, i.e., "it feels locally Euclidean". Such objects are called **manifolds** and are among the most commonly studied spaces in computational topology.

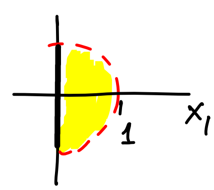
**Def** A topological space $\mathbb{M}$ is a **2-manifold** if all points in $\mathbb{M}$ lie in open discs, i.e., every point has a neighborhood $\approx D = \{\bar{x} \in \mathbb{R}^2 \mid \|x\| < 1\}$.

e.g., $\mathbb{S}^2$, $\mathbb{R}^2$. these are 2-manifolds without boundary.

A **2-manifold with boundary** is a topological space $\mathbb{M}$ whose every point has a neighborhood homeomorphic to $D$ or to $D_+ = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1, x_1 \geq 0\}$ (but not both), and there exist some points of the latter type.

$1^{st}$ entry



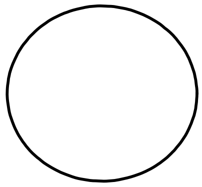
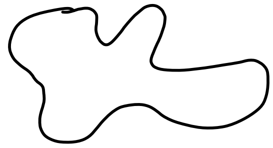
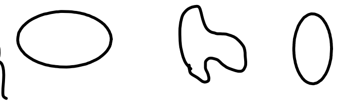



points on the boundary have neighborhoods that are half discs.

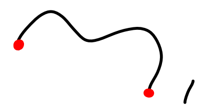
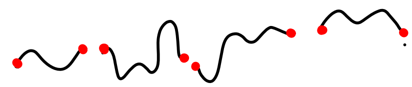
**Def** The **boundary** of a 2-manifold with boundary $\mathbb{M}$ is the set of points in $\mathbb{M}$ that have neighborhoods homeomorphic to the half disc.

(The definition of $\partial A$ used for sets $A$ is equivalent to this definition).

Notice that the 1-manifold is just the circle ($\mathbb{S}^1$),

 or  or {},

a collection of disjoint circles.

A 1-manifold with boundary: , or {}

only one end point is included here!

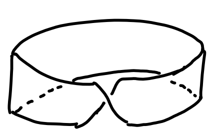
boundary is indeed the set of end points.

0-manifold: Any collection of distinct points (discrete set of points).

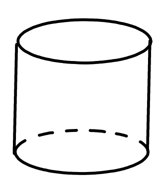
2-manifolds, also called surfaces, are a well studied class of spaces (objects), both from the theoretical as well as applied points of view. We will present several details of the properties of 2-manifolds first. To define and study d-manifolds for $d \geq 2$, we will need a few more definitions and concepts from analysis/point set topology.

By default, we assume a manifold (w/ or w/o boundary) is connected.

A 2-manifold (for that matter, d-manifold for $d \geq 2$) is orientable or non-orientable.



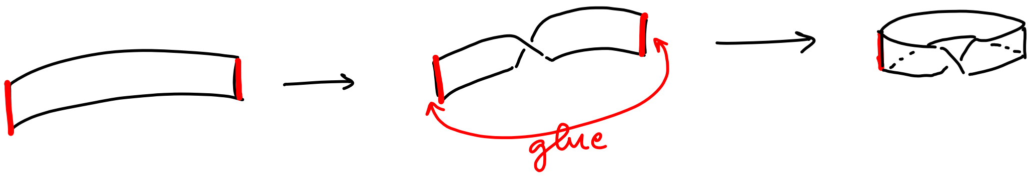
Möbius strip is non-orientable

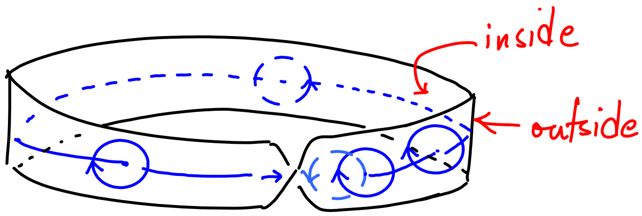


cylinder is orientable

The Möbius strip has only one "side," while the cylinder has two "sides" — inside and outside.

We obtain the Möbius strip by taking a rectangular strip of paper, and gluing its short edges together after twisting the strip once.




there are two "sides" at each point on the Möbius strip – front/back or inside/outside.

Consider sliding an oriented loop, or a clock along the surface of the Möbius strip. Look at the path followed by the center of the clock. Once the center comes back to where it started, its orientation is reversed (as it goes over the "twist").

The path traced by the center of the clock here is hence an **orientation reversing** closed curve. If the orientation is not reversed this way, the curve is said to be **orientation preserving**.

**Def** If all closed curves in a 2-manifold (with or without boundary) are orientation preserving, then the 2-manifold is **orientable**, else it is **nonorientable**.

$\mathbb{R}^2$, $S^2$, torus, etc.

Möbius strip, Klein bottle, etc.

# Classification of Manifolds

Enumerate all possible manifolds upto homeomorphisms for each dimension.

We mention the classification for 0- and 1-dimensional manifolds, but will come back to give some more definitions to finish the discussion for 2-manifolds.

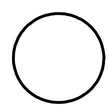
each point has to have a neighborhood $\approx \mathbb{R}^0$, i.e., a point.

<u>0-manifolds</u>: a discrete space, e.g., $\mathbb{Z}^2$ – all points with integer coordinates in $\mathbb{R}^2$.

Notice that $2\mathbb{Z}^2$, all points with even integer coordinates in $\mathbb{R}^2$ is homeomorphic to $\mathbb{Z}^2$.

We will define "compact" soon...

<u>1-manifolds</u>
connected

- compact: $\approx \mathbb{S}^1$, 
- compact with boundary: $\approx [0,1]$
  closed unit interval
- non-compact: $\approx \mathbb{R}^1$ or $(0,1)$
- non-compact with boundary $\approx \mathbb{R}^1_+$ or $[0,1)$
  $x \geq 0$
  half-open unit interval