MATH 529 — Lecture 5 (01/23/2024)

Today: * classification of 2-manifolds
* simplices and simplicial complexes
* abstract simplicial complexes

Classification of Compact, connected 2-manifolds

Result Every compact, connected 2-manifold is homeomorphic to $\mathbb{S}^2$, or a connected sum of copies of $\mathbb{T}^2$, or a connected sum of copies of $\mathbb{RP}^2$.
If a 2-manifold is not connected, each component has this structure.

Examples

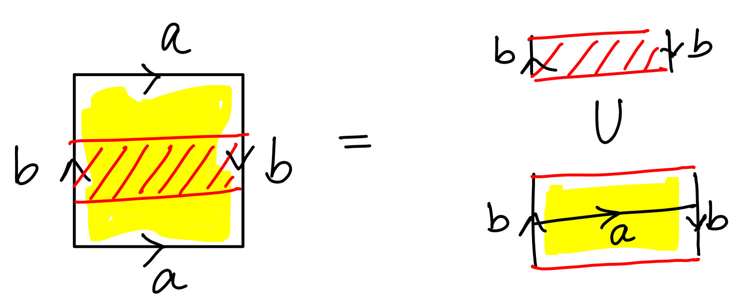
1. $\mathbb{S}^2 \# \mathbb{RP}^2 \approx \mathbb{RP}^2$

you just close back the open disc cut out from $\mathbb{M}$!

In fact, $\mathbb{S}^2 \# \mathbb{M} \approx \mathbb{M}$ for any 2-manifold $\mathbb{M}$.

2. $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx \mathbb{K}^2$ Here's a "proof" by pictures:

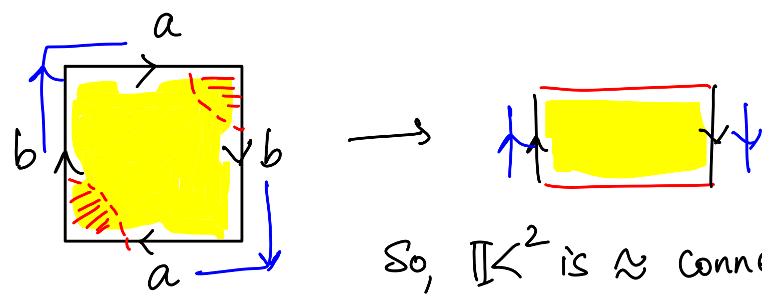
$\mathbb{K}^2$:



Connected sum of 2 Möbius strips!

And $\mathbb{RP}^2 - D^2$:

open 2-disc



Mobius strip

So, $\mathbb{K}^2$ is $\approx$ connected sum of 2 $\mathbb{RP}^2$'s.

Also, we get $\mathbb{RP}^2$ as $\mathbb{S}^2 \# M$, for a Möbius strip M.

3. $\mathbb{T}^2 \# \mathbb{RP}^2 \approx \#(\mathbb{RP}^2)^3$

orientable nonorientable

Once you join a non-orientable 2-manifold, the result stays non-orientable.

As one would expect, the corresponding result for 3-manifolds is much more complicated. Thurston's geometrization conjecture states that all compact 3-manifolds can be canonically decomposed into submanifolds that have geometric structure. This conjecture implies the famous Poincaré conjecture, which states that every compact, simply connected 3-manifold is homeomorphic to $\mathbb{S}^3$, the 3-sphere.

(Informally, an object is simply connected if there are no "holes" passing through the object).

Perelman presented a proof of the Poincaré conjecture, almost 100 years after it was originally proposed (in 1904; Perelman's proof appeared in 2003). The corresponding result for n-manifolds with $n \geq 4$ turns out to be easier to prove, informally because of the "increased geometric freedom" one can afford in higher dimensions.

The main concepts used in Perelman's proof can be used to provide a proof for Thurston's geometrization conjecture (Perelman presented such a proof in 2003, along with his proof of the Poincaré conjecture).

Simplices

While we can study sample 2-manifolds as is, we cannot do computations on them. For this purpose, we need a discretized version of the spaces in question, which could be stored and handled naturally by a computer. Can we, for instance, use some sort of "counting arguments" to distinguish S^2 from $\mathbb{T}^2$?

We introduce the concept of simplicial complexes in this context, and use concepts from combinatorial algebraic topology. The idea is that we can handle the combinatorics using efficient algorithms. Similarly, there are efficient data structures that can be used to work with simplicial complexes modeling the spaces. We also want to separate the topology from the geometry of the object. we use the geometry in many applications.

We first introduce some definitions.

Def (Combinations) Let $S = \{\bar{p}_0, \dots, \bar{p}_k\} \subseteq \mathbb{R}^d$. A **linear combination** of $\bar{p}_i$ is $\bar{x} = \sum_{i=0}^{k} \lambda_i \bar{p}_i$, $\lambda_i \in \mathbb{R}\ \forall i$.

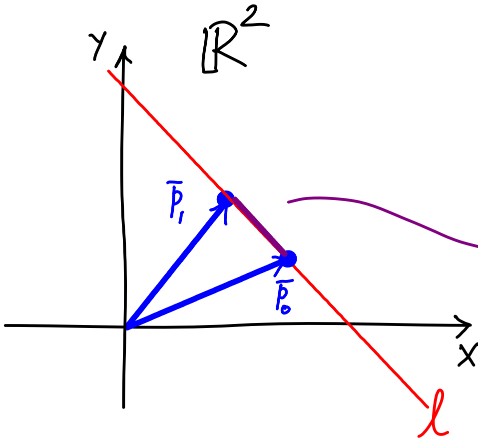
If $\sum_{i=0}^{k} \lambda_i = 1$, then $\bar{x}$ is an **affine combination** of $\bar{p}_i$'s.

In addition, if $\lambda_i \geq 0\ \forall i$, $\bar{x}$ is a **convex combination** of $\bar{p}_i$'s.

The set of all convex combinations of elements in S is the **convex hull** of S, denoted as

$$\operatorname{conv}(S) = \left\{ \sum_{i=0}^{k} \lambda_i \bar{p}_i \;\middle|\; \lambda_i \geq 0\ \forall i,\ \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Illustration in 2D



set of all linear combinations $= \mathbb{R}^2$

set of all affine combinations $= \ell$, the line through $\bar{p}_0, \bar{p}_1$

$\text{conv}(\{\bar{p}_0, \bar{p}_1\}) =$ line segment connecting $\bar{p}_0, \bar{p}_1$.

$\bar{p}_0, \bar{p}_1$ are not parallel here.

Def (Independence) S with $|S| \geq 2$ is **linearly (affinely) independent** if no point in S is a linear (affine) combination of other points in S.

We denote linearly independent in short as LI, and affinely independent as AI.

$|S|=1$ case: $\{\bar{p}_0\}$ is LI if $\bar{p}_0 \neq \bar{0}$ (zero vector), but $\{\bar{p}_0\}$ is AI for all $\bar{p}_0$ (even if $\bar{p}_0 = \bar{0}$).

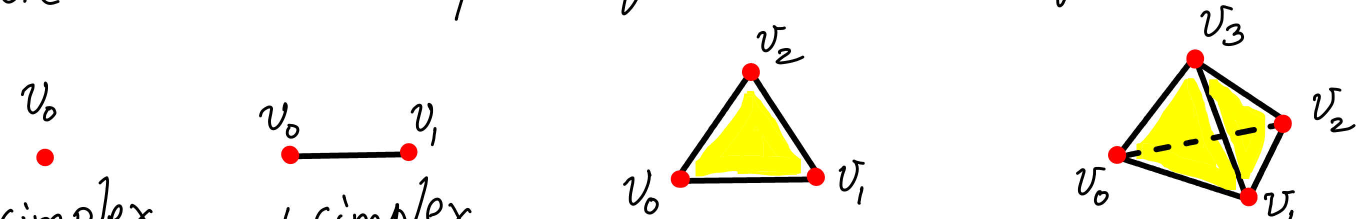
For example, 3 points in $\mathbb{R}^2$ are AI as long as they are not collinear.

We will use convex hulls of AI points as our building blocks — called simplices.

Def (simplex) The convex hull of $(k+1)$ affinely independent points $S = \{\bar{v}_0, \dots, \bar{v}_k\}$ is a **k-simplex**. The **dimension** of the simplex is k, and $\bar{v}_j$'s are the **vertices** of the k-simplex.

vertices, and hence $\bar{v}_i$, rather than $\bar{P}_i$.

Here are the k-simplices for small values of k: $0 \le k \le 3$.



0-simplex (point) | 1-simplex (line segment) or edge | 2-simplex (triangle) | 3-simplex (tetrahedron)

Notice that the k-simplex is homeomorphic to the k-ball, i.e., $B_k = \{\bar{x} \in \mathbb{R}^k \mid \|\bar{x}\| \le 1\}$. "=" gives S^{k-1}, the $(k-1)$-sphere.

Indeed, the boundary of the k-ball is the $(k-1)$-sphere, e.g., 2-ball (or 2-disc) has the circle (1-sphere) as the boundary.

Each p-simplex is made of lower dimensional simplices, i.e., k-simplices with $k \le p$. Thus, $\Delta v_0 v_1 v_2$ contains vertices v_0, v_1, v_2, edges $\overline{v_0 v_1}, \overline{v_1 v_2}, \overline{v_0 v_2}$, and $\Delta v_0 v_1 v_2$ itself.

Def (face/coface). Let σ be the k-simplex defined on $S = \{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_k\}$. A simplex τ defined on a subset T of S, $|T| \neq \phi$, is a **face** of σ, and σ is a **coface** of τ. The notation is $\tau \leq \sigma$, $\sigma \geq \tau$ (some books use $\tau \preceq \sigma$, $\sigma \succeq \tau$ \succeq, \preceq in LaTeX.

Thus, $\overline{v_0 v_1}$ is a face of $\Delta v_0 v_1 v_2$. So are v_0, v_1, v_2, $\overline{v_1 v_2}$, and $\overline{v_0 v_2}$.

A simplex is always a face of itself, i.e., $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

We can attach simplices together "nicely" to form bigger objects called simplicial complexes.

Def A **simplicial complex** K is a set of simplices such that

1. $\sigma \in K$, $\tau \leq \sigma \Rightarrow \tau \in K$; and

   Every face of a simplex in K is also in K.

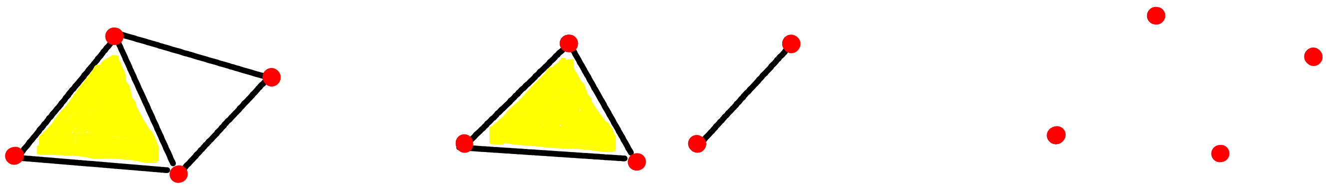
2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ when $\sigma \cap \sigma' \neq \phi$.

   In particular, the non-empty intersection of two simplices in K is a face of both of them, and hence in K as well.

The above definition holds in the case of both finite and nonfinite K. In the latter case, K has infinitely many simplices satisfying the two conditions. But we will usually limit our attention in this course to finite simplicial complexes, unless mentioned otherwise.

Note that a simplex is the convex hull of a **finite** number of affinely independent points. So, we never talk about "infinite-dimensional" simplices. We could talk about each of these points sitting in infinite-dimensional space, but we restrict our attention to $\mathbb{R}^d$ for finite d.

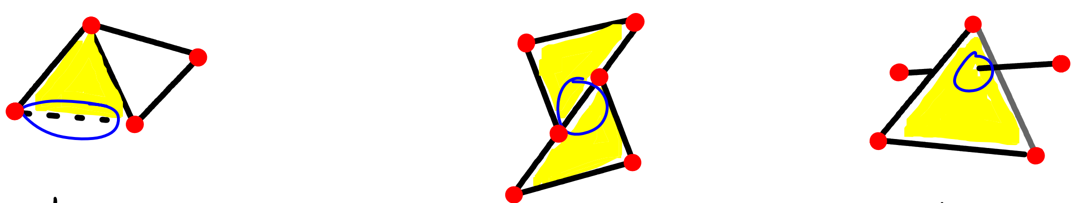
Examples Here are some simplicial complexes:



K_1 $\qquad$ K_2 $\qquad$ K_3

In particular, notice that a simplicial complex need not consist of just one connected component.

Here are some collections that are not simplicial complexes:



K_4 $\qquad$ K_5 $\qquad$ K_6

K_4 violates Condition 1, as one of the faces of the triangle in K_4 is not in the collection. K_5 and K_6 violate Condition 2, as the intersection of two simplices is not a face of either simplex in both cases.

Def The dimension of a simplicial complex is the same as that of the highest dimensional simplex in it, i.e.,

$$\dim K = \max \{\dim \sigma \mid \sigma \in K\}.$$

In the previous examples, $\dim(K_1) = 2$, $\dim(K_2) = 2$, and $\dim(K_3) = 0$.

Earlier we have been talking about continuous surfaces, e.g., the 2-sphere, torus, etc. And now we are talking about simplicial complexes as discrete objects. How do we "reconcile" the two notions? Indeed, we can formally define the "space" modeled by a simplicial complex. Later on, we will talk about a simplicial complex "triangulating", say, a torus, when this "space" is homeomorphic to $\mathbb{T}^2$.

Def The **underlying space** of a simplicial complex K the space made of all simplices in K together with the topology inherited from the ambient Euclidean space. We denote the underlying space of the simplicial complex K by $|K|$. Thus,

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

$|K|$ is also called the polyhedron (or polytope) of K.

$A \subseteq |K|$ is closed in $|K|$ iff $A \cap \sigma$ is closed $\forall \sigma \in K$.

We can define simplicial complexes abstractly – simplices need not necessarily be sitting in $\mathbb{R}^d$ for any d. Indeed, this notion conveys the power and versatility of the concept!

Def An **abstract simplicial complex** (ASC) is a collection $\mathcal{S}$ of finite non-empty sets such that if $A \in \mathcal{S}$, and $B \subseteq A$ with $B \neq \emptyset$, then $B \in \mathcal{S}$.

Note that the condition specified in the above definition of an abstract simplicial complex is equivalent to the first condition in the definition of a (regular) simplicial complex, which says that every face of a simplex in the complex is also in the complex.

The second intersection condition is trivially satisfied in the case of abstract simplicial complexes. The intersection of two sets is indeed a subset of both sets.
$\mathcal{S}$ itself can be finite or infinite, but each $A \subseteq \mathcal{S}$ is a finite set.

The sets in $\mathcal{S}$ are called the **simplices** of $\mathcal{S}$. The dimension of a simplex $A \in \mathcal{S}$ is $\dim(A) = |A| - 1$.

cardinality (# entries) of A

Note the correspondence of the above definition to the definition of simplices in the usual sense. Recall that a k-simplex is the convex hull of $(k+1)$ affinely independent points, which are its vertices.
We maintain this correspondence by defining $\dim A = |A| - 1$ for any set $A \in \mathcal{S}$.

The **dimension** of $\mathcal{S}$ is $\dim(\mathcal{S}) = \max\{\dim(A) \mid A \in \mathcal{S}\}$.

In the definition of $\mathcal{S}$, we do assume that all $A \in \mathcal{S}$ are finite sets. And there exists a maximum dimensional simplex in $\mathcal{S}$.

which is finite

The singleton sets in $\mathcal{S}$ are called its **vertices**, and is denoted by Vert $(\mathcal{S})$.

Again note the correspondence of these singleton sets to the vertices (0-simplices) in (geometric) simplicial complexes.

Here is an example of an abstract simplicial complex.

$$\mathcal{S} = \{\underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}\}.$$

We can indeed check that the condition on inclusion of subsets is satisfied. For instance, consider the set $\{0,1,2\}$.

$$\{0,1,2\} \in \mathcal{S} \Rightarrow \text{need } \underset{\text{trivial}}{\{0,1,2\}}, \underbrace{\{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\}}_{\text{indeed!}} \in \mathcal{S}.$$

$\dim \mathcal{S} = 2.$