MATH 529 – Lecture 6 (01/25/2024)

Today: * geometric realization of ASCs
* Comparing ASCs
* topological invariants

Recall: ASC $\mathcal{S}$; $A \in \mathcal{S}$, $B \subset A$, $B \neq \emptyset \Rightarrow B \in \mathcal{S}$.

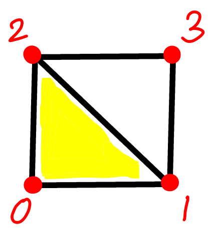
The singleton sets in $\mathcal{S}$ are called its **vertices**.

Given any (geometric) simplicial complex K, we can create an abstract simplicial complex $\mathcal{S}$ by taking just the sets of vertices in each simplex of K (and ignoring the geometry). $\mathcal{S}$ here is called the **vertex scheme** of K.

Symmetrically, K is a **geometric realization** of $\mathcal{S}$.
there could be other geometric realizations

$$\mathcal{S} = \{ \underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\} \}.$$

K



K can sit in $\mathbb{R}^2$

We have a vertex corresponding to each singleton set (i.e., vertex) in $\mathcal{S}$, an edge corresponding to each doublet, a triangle for each triplet, etc.

But this is just one geometric realization. In particular, if we were to specify the (x,y) coordinates of the vertices, we can imagine other realizations, e.g., by translating this one. We could also have a realization in $\mathbb{R}^3$, for instance.

Theorem (Geometric realization theorem) Every abstract simplicial complex $\mathcal{S}$ with $\dim \mathcal{S} = d$ has a geometric realization in $\mathbb{R}^{2d+1}$.

Idea of the Proof

set of vertices in $\mathcal{S}$

We map vertices of $\mathcal{S}$ injectively to points in $\mathbb{R}^{2d+1}$, say, $f : \text{Vert}(\mathcal{S}) \to \mathbb{R}^{2d+1}$. Why $2d+1$? We use the fact that $2d+2$ or fewer points in $\mathbb{R}^{2d+1}$ that are in **general position** are affinely independent (AI).

Def $(d+1)$ points in $\mathbb{R}^d$ are in **general position** if no hyperplane contains more than d of those points.

The idea is that the points do not satisfy any more linear relationships than they must. For instance 3 points in $\mathbb{R}^2$ that are not collinear are in general position.

Recall that a d-simplex is the convex hull of $(d+1)$ AI points. We need to make sure that we will have "enough freedom", i.e., affine independence, among the mapped vertices so that we can map all the simplices in $\mathcal{S}$ to corresponding simplices in the geometric simplicial complex.

Consider $A, B \in \mathcal{S}$. Since $\dim \mathcal{S} = d$, $|A|, |B| \leq d+1$.

Hence, $|A \cup B| = |A| + |B| - |A \cap B| \leq d+1 + d+1 = 2d+2$.

Hence by going to $\mathbb{R}^{2d+1}$ and choosing points there in general position, we can ensure that (up to) $2d+2$ points are AI.

$\Rightarrow$ Any convex combination $\bar{x}$ in $A \cup B$ is unique.

$\Rightarrow$ $\bar{x} \in A$ and $\bar{x} \in B \iff \bar{x} \in A \cap B$.

$\hookrightarrow$ ensures the second requirement of nonempty intersections of two simplices being their faces.

We could often find geometric realizations in $\mathbb{R}^{d'}$ for d' smaller than $2d+1$.

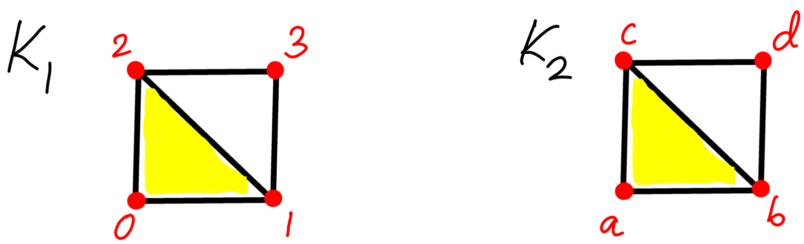
$\square$

How do we compare abstract simplicial complexes? We had previously defined the concept of homeomorphism to study when two topological spaces are "similar". We now define corresponding notions for simplicial complexes. Recall that $\mathrm{Vert}(\mathcal{S})$ represents the vertex set of the ASC $\mathcal{S}$.

Def Two abstract simplicial complexes $\mathcal{S}_1$ and $\mathcal{S}_2$ are **isomorphic** if there exists a bijection $\varphi : \mathrm{Vert}(\mathcal{S}_1) \to \mathrm{Vert}(\mathcal{S}_2)$ such that $A \in \mathcal{S}_1$ iff $\varphi(A) \in \mathcal{S}_2$. φ is an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$. We write $\mathcal{S}_1 \approx \mathcal{S}_2$ here.

simplex

In this setting, every simplex in $\mathcal{S}_1$ has a unique corresponding simplex in $\mathcal{S}_2$.



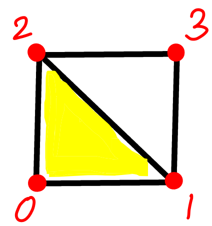
vertex schemes of K_1 and K_2 are isomorphic

Notice that the similarity here is defined for **abstract** simplicial complexes. Hence, K_1 and K_2 above need not be sitting in the same space. Still, they are isomorphic as ASCs. We make this notion precise in the following theorem.

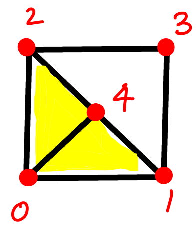
Theorem Two (geometric) simplicial complexes K_1 and K_2 are isomorphic, or **simplicially homeomorphic**, iff their vertex schemes $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic as abstract simplicial complexes. We denote this fact by $K_1 \cong K_2$, which implies $|K_1| \approx |K_2|$, and $\mathcal{S}_1 \approx \mathcal{S}_2$.

The implication might not go the other way, though. For instance, $K_1 \cong K_2$ above. Now consider K_3 as shown below.

K_1



K_3



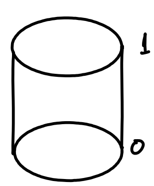
Notice that $K_1 \ncong K_3$, even though $|K_1| \approx |K_3|$. In fact, their underlying spaces could very well be identical!

From the computational point of view, while simplicial complexes that are "smaller", i.e., have a smaller number of simplices while modeling the same topological space are usually preferred. At the same time, geometry might dictate that we need a large number of simplices to capture the complexity.

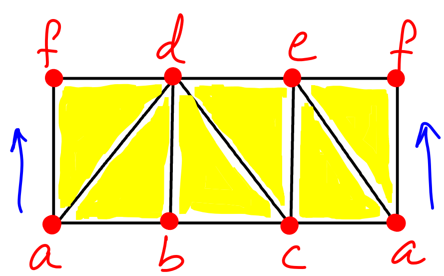
How do we use ASCs? We illustrate several examples

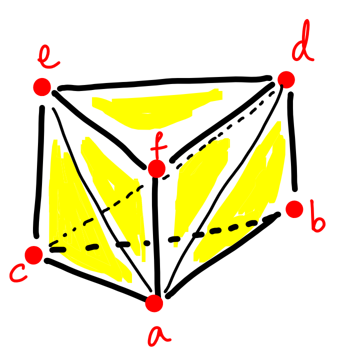
1. cylinder $S^1 \times [0,1]$

geometric representation!



L





K

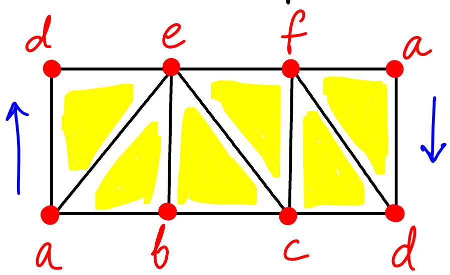
The underlying space of L appears to be rectangle, but notice how the vertex labels identify the left and right edges (both are af).

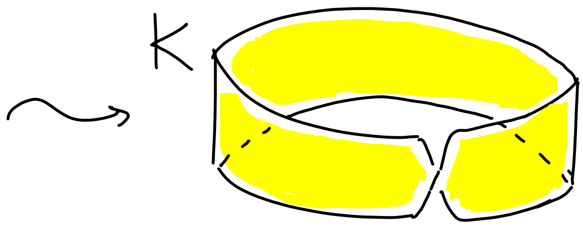
$$L = \{\{a,b,d\}, \{a,d,f\}, \{b,c,d\}, \{c,d,e\}, \{a,c,e\}, \{a,e,f\}, \text{ and all nonempty subsets}\}$$

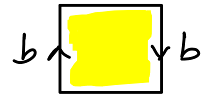
K is one geometric realization of L here.

2. Möbius strip — We start with the ASC L here

L







The left and right vertical edges are identified, after a "twist". Like in Example 1, the underlying space is a rectangle, but vertex labels are different.

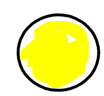
L represents the Möbius strip, i.e., K is a geometric realization of L.

We could also specify L abstractly:

$L = \{ \{a,d,e\}, \{a,b,e\}, \{b,c,e\}, \{c,e,f\}, \{c,d,f\}, \{a,d,f\}$, and all nonempty subsets of these triplets $\}$.

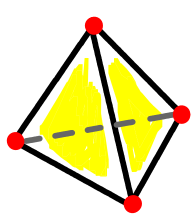
The abstract simplicial complexes shown above consist of triangles — indeed, they are triangulations. In general, triangulations consist of triangles (in 2D), and simplices in general as we formalize below.

Notice that each k-simplex $\approx$ k-ball (closed). 2-ball is the closed 2-disc.



Def (Triangulation). A **triangulation** of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.

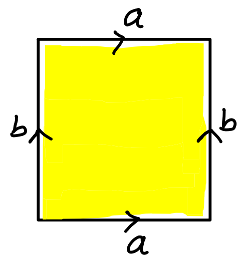
Example:



surface of a tetrahedron (triangles and their faces) is a triangulation of the 2-sphere S^2.

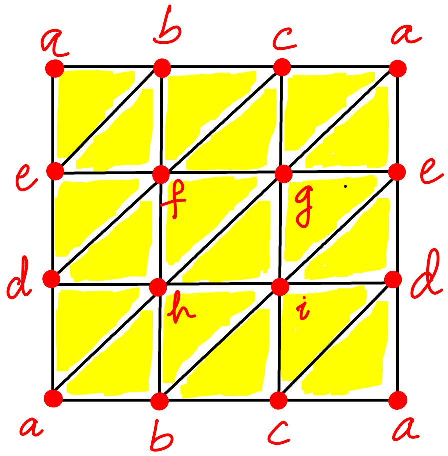
A triangulation is a piecewise linear representation of the topological space.

3. Torus ($\mathbb{T}^2$)



$\longrightarrow$ this diagram is not a triangulation!

K:

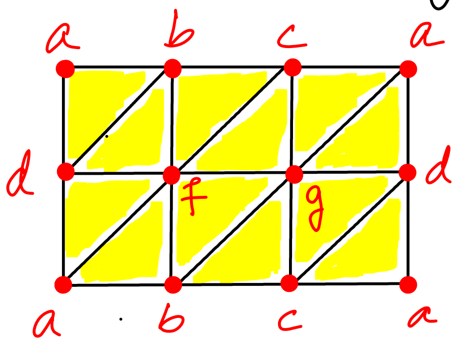


Why so many triangles?

This triangulation K of the torus has 18 (different) triangles. One might wonder if we could produce a triangulation using a much smaller number of triangles.

Consider the following candidate triangulation K':

K'



Is K' a triangulation of $\mathbb{T}^2$? No! For instance, consider edge $\overline{ad}$. It is a face of four triangles: adb, adf, adc, adg.

Hence, points on $\overline{ad}$ do not have neighborhoods homeomorphic to $\mathbb{R}^2$. We're doing "too much gluing" here.

Q. What is the minimum number of triangles needed to produce a triangulation of $\mathbb{T}^2$?

A. We need at least 14 triangles.

Rule: In a triangulation of a 2-manifold (with boundary), each edge must be part of (one or) two triangles.

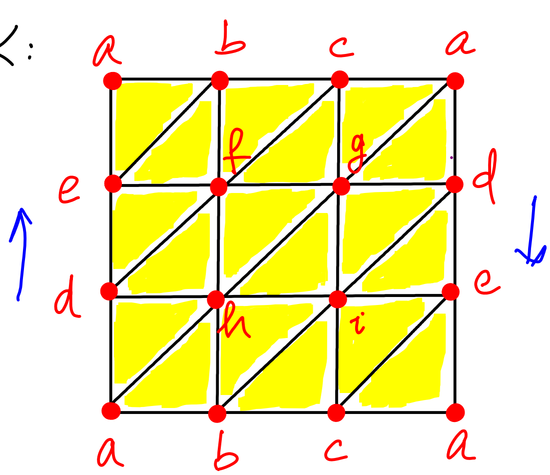
Edges that are part of only one triangle each form the boundary of the 2-manifold.

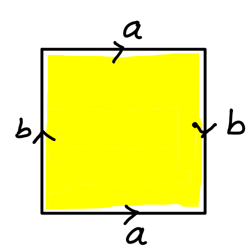
Recall that every point on a 2-manifold has a neighborhood homeomorphic to an open disc. The corresponding requirement for edges becomes that each edge has to be part of exactly two triangles. Similarly, points on the boundary of a 2-manifold have neighborhoods homeomorphic to half discs. Edges that are part of single triangles are indeed the boundary edges in the triangulation. The result extends to d-dimensions — every $(d-1)$-simplex is the face of one or two d-simplices.

The above rule could be used to check if a given simplicial complex is the triangulation of a manifold or not. But, be warned that satisfying this rule **alone** is not enough to identify the given simplicial complex as the triangulation of a specific manifold, e.g., the torus.

4. Klein bottle ($\mathbb{K}^2$)

K:





Same rule applies here — each edge is shared by two triangles exactly.

For example, edge $\overline{ae}$ is part of triangles Δabe and Δace.

We talked about distinguishing spaces that are not homeomorphic. In practice, spaces or objects are usually represented by triangulations. Checking for homeomorphisms between triangulations is not easy. How can we distinguish two topological spaces computationally? One option is to use an invariant.

Topological Invariants

Def A **topological invariant** is a map that assigns the same object to spaces of the same topological type.

↳ usually, a number; but we could also have a "barcode", for instance

Let $f(\cdot)$ be an invariant.

$\mathbb{X} \approx \mathbb{Y} \Rightarrow f(\mathbb{X}) = f(\mathbb{Y})$.

So, $f(\mathbb{X}) \neq f(\mathbb{Y}) \Rightarrow \mathbb{X} \not\approx \mathbb{Y}$ → could be used for contrapositive arguments.

But $f(\mathbb{X}) = f(\mathbb{Y})$ does not necessarily mean $\mathbb{X} \approx \mathbb{Y}$.

If $f(\mathbb{X}) = f(\mathbb{Y}) \Rightarrow \mathbb{X} \approx \mathbb{Y}$, then $f(\cdot)$ is called a **complete invariant**.

Notice that an invariant could assign the same object to spaces of different topological types. The main way to use the invariant is in the contrapositive, i.e., if the invariant is different for a pair of spaces, then the two spaces have different topological types.