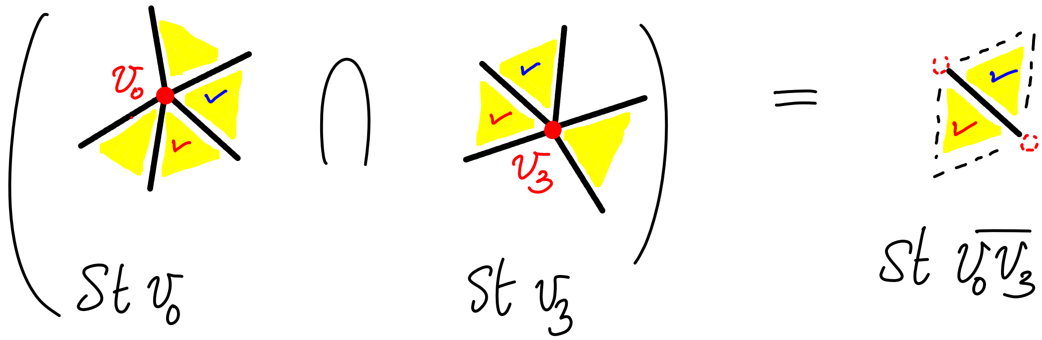
MATH 529: Lecture 10 (02/12/2026)

Today:
* star of $X \subset K$
* poset representation
* retraction, homotopy equivalence

Recall: $\operatorname{St} \bar{v} = \bigcup_{\sigma \geq v} \operatorname{Int} \sigma$, $\quad \operatorname{St} \sigma = \bigcup_{\tau \geq \sigma} \operatorname{Int} \tau$. How are these two concepts related?

For $\sigma = [v_0, \dots, v_k]$, $\quad \operatorname{St} \sigma = \left(\bigcap_{i=0}^{k} \operatorname{St} v_i \right)$

For instance, with $\sigma = v_0 v_3$, we get



Def For $X \subseteq K$, we define

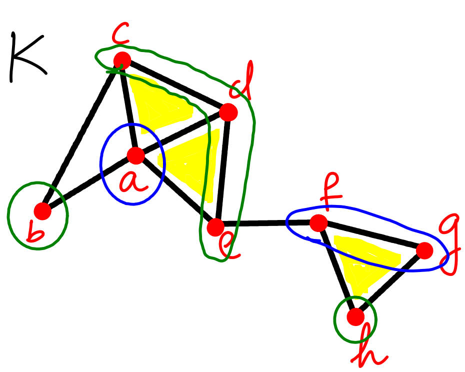
$$\operatorname{St} X = \left\{ \bigcup \tau \mid \tau \geq \sigma, \sigma \in X \right\}.$$

Set of cofaces of each simplex in the set X.

$\operatorname{Cl} \operatorname{St} X$ is the closure of $\operatorname{St} X$. We define

$\operatorname{Lk} X =$ Set of simplices in $\operatorname{Cl} \operatorname{St} X$ that do not belong to $\operatorname{St}(\bar{X})$.

The intuition of the star being the open neighborhood of X and link being the boundary of the closed neighborhood still holds.



Let $X = \{a, \overline{fg}\}$. Then we intuitively want

$$\operatorname{Lk} X = \{b, c, d, e, \overline{cd}, \overline{de}, h\}.$$

We follow the definitions to get the following sets.

$$\mathrm{St}\,X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh\}.$$

$$\mathrm{Cl}\,\mathrm{St}\,X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh, \ldots \; b, c, d, e, \overline{cd}, \overline{de}, \cancel{f}, \cancel{g}, h, \cancel{\overline{fh}}, \cancel{\overline{gh}}, \}$$

the simplices added as proper faces

We also get

$$\mathrm{Cl}\,X = \{a, \overline{fg}, f, g\}, \text{ and}$$

faces added to close X

$$\mathrm{St}\,\mathrm{Cl}\,X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh, \ldots \; \underline{f, g, \overline{ef}, \overline{fh}, \overline{gh}}\}$$

cofaces of the elements added to close X

$$\Rightarrow \mathrm{Lk}\,X = \mathrm{Cl}\,\mathrm{St}\,X - \mathrm{St}\,\mathrm{Cl}\,X = \{b, c, d, e, \overline{cd}, \overline{de}, h\},$$

as expected!

Note that $\overline{ef} \in \mathrm{St}\,\mathrm{Cl}\,X$, but is not in $\mathrm{Lk}\,X$ (as per definition).

For a small example, we can easily eye-ball these sets. But how do you handle large simplicial complexes with, say, 10^4 simplices?

We describe a way to efficiently store simplicial complexes and to read off St X, Lk X, Cl X, etc. from that representation.

Def (poset) Given a finite set S, a **partial order** is a binary relation $\leq$ on S that is reflexive, antisymmetric, and transitive, i.e., $\forall\, x,y,z \in S$,

partial, as not every $x,y \in S$ are related by $\leq$.

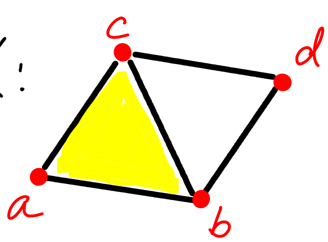
(a) $x \leq x$;

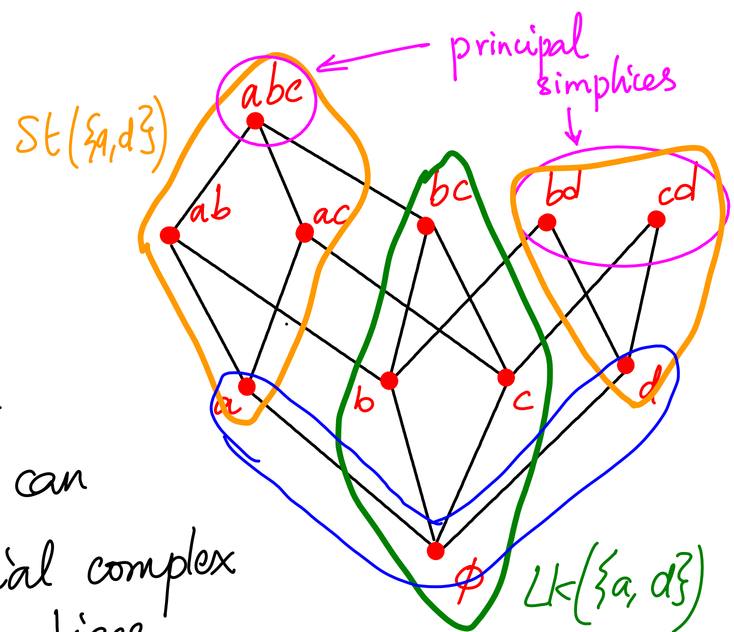
(b) $x \leq y$ and $y \leq x \Rightarrow x = y$; and

(c) $x \leq y,\ y \leq z \Rightarrow x \leq z$.

A set S with a partial order is called a partially ordered set, or a poset.
The face relationships of a simplicial complex is a partial order. So the vertex scheme of a simplicial complex with face relationships is a poset.

Illustration

K:




The simplices "at the top" are called principal simplices. We can determine the entire simplicial complex if we know the principal simplices.

To find Star ($St\,X$), take X and everything above. For instance,

$St(\{a,d\}) = \{a, ab, ac, abc, d, bd, cd\}$.

To find $Cl\,X$, take X and everything below; e.g., $Cl(\{a,d\}) = \{a, d, \emptyset\}$.

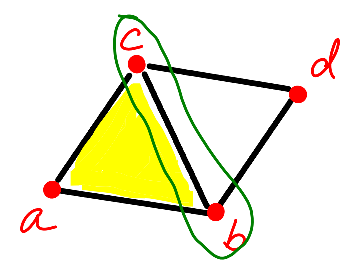
Notice that $Cl\,St(\{a,d\}) = K \cup \{\emptyset\}$ here.

As a convention, the empty simplex (or null set) is added at the bottom of this poset representation. It plays the role of the "root node" from which the poset representation "grows up".

Hence, we include the empty set $\emptyset$ in our definitions and discussions of closure, star, and link. In particular, we modify the definition of link slightly as follows:

$$Lk\,X = Cl\,St\,X - St(Cl\,X - \{\emptyset\}).$$

With $X = \{a, d\}$, we expect $Lk\,X$ to be $\{bc, b, c\}$.



Recall, $Cl\,X = \{a, d, \emptyset\}$. So,

$St(Cl\,X - \{\emptyset\}) = St(\{a,d\})$ here. Hence we indeed get

$$Lk\,X = \{bc, b, c, \emptyset\}.$$

We now define a notion of topological similarity that is weaker than homeomorphism. We then use this notion to define how to build simplicial complexes on data sets of points in $\mathbb{R}^d$.

Homotopy

Def Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous maps from topological space $\mathbb{X}$ to space $\mathbb{Y}$. A **homotopy** between f and g is another continuous map

$H : \mathbb{X} \times \underbrace{[0,1]}_{t} \longrightarrow \mathbb{Y}$ such that H agrees with f at $t=0$, and with g at $t=1$. In other words,

$$H(x,0) = f(x) \ \forall x \in \mathbb{X}, \text{ and}$$
$$H(x,1) = g(x) \ \forall x \in \mathbb{X}.$$

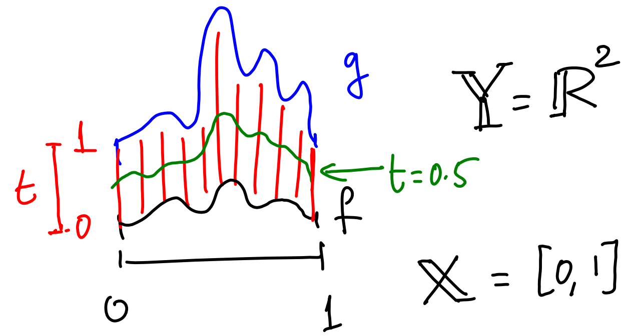
The index t can be thought of as time, varying from 0 to 1.

H could be thought of a time-series of functions $f_t : \mathbb{X} \to \mathbb{Y}$ for $t \in [0,1]$, with $f_t(x) = H(x,t)$, where $f_0 = f$ and $f_1 = g$.

We say that f is **homotopy equivalent** to g, or that f is homotopic to g. We denote this equivalence relation by $f \simeq g$.

reflexive, symmetric, and transitive.

Here is an illustration, with $\mathbb{X} = [0,1]$ and $\mathbb{Y} = \mathbb{R}^2$. The homotopy H is a 2D strip of functions going from f to g. All of f, g, and H are continuous.

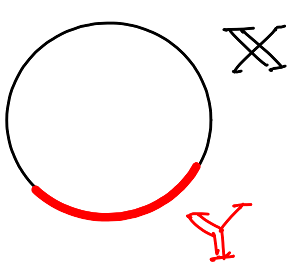


We extend the definition of homotopy to topological spaces.
First we consider a special case.

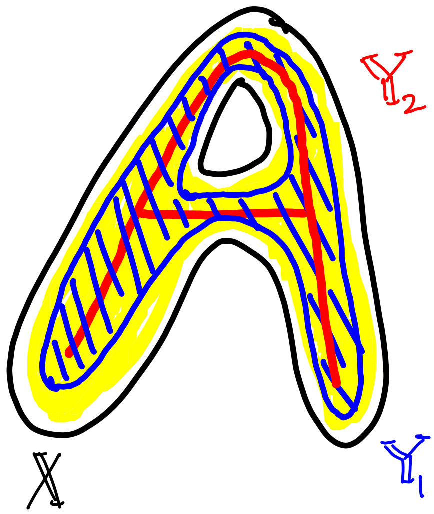
Def $\mathbb{Y} \subseteq \mathbb{X}$ is a **retract** of $\mathbb{X}$ if there is a continuous map $r : \mathbb{X} \to \mathbb{Y}$ with $r(y) = y \ \forall y \in \mathbb{Y}$.
r is called a **retraction**.

Def $\mathbb{Y}$ is a **deformation retract** of $\mathbb{X}$, and r is a **deformation retraction**, if there is a homotopy between the retract r and the identity map $id_{\mathbb{X}}$ on $\mathbb{X}$, i.e., $r \simeq id_{\mathbb{X}}$.
$id_{\mathbb{X}}(x) = x \ \forall x \in \mathbb{X}$.

We also say that $\mathbb{X}$ deformation retracts to $\mathbb{Y}$.

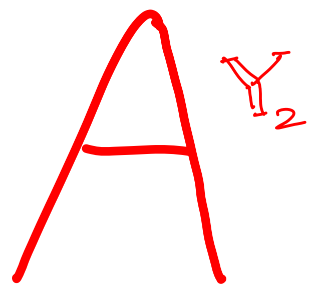


Here is an example of a retract that is not a deformation retract.
Notice that $\mathbb{X}$ is S^1 (circle), while $\mathbb{Y}$ is just an open arc.



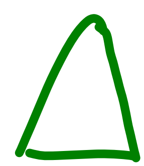
$\mathbb{Y}_1$ is a deformation retract of $\mathbb{X}$.

Continue to deform to obtain



$(\mathbb{Y}_2 \subset \mathbb{X})$.
"skeleton" sitting inside the "fat A".

$\mathbb{Y}_2 \not\approx \mathbb{X}$, but $\mathbb{Y}_2$ and $\mathbb{X}$ have the same homotopy type.
we'll define it formally soon!

Deforming even further, we can get $\mathbb{Y}_3$  $(\mathbb{Y}_3 \subset \mathbb{Y}_2)$.

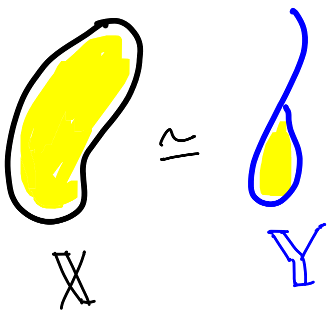
$\mathbb{X}$, and $\mathbb{Y}_j$, $j = 1, 2, 3$ are all homotopy equivalent. Also, each $\mathbb{Y}_j$ is a deformation retract of $\mathbb{Y}_k$ for $k < j$, and also of $\mathbb{X}$.

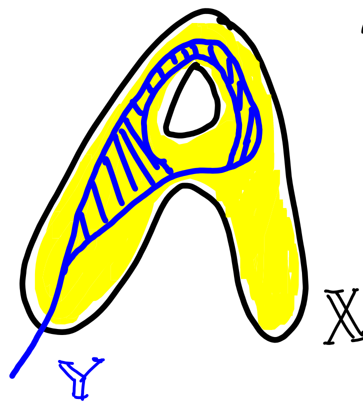
Notice that while $\mathbb{X}$ and $\mathbb{Y}_2$, for instance, are not homeomorphic, they both are forms of the letter 'A'. $\mathbb{Y}_2$ is, in some sense, the "skeleton" of $\mathbb{X}$. These types of transformations are allowed in the less tight notion of topological similarity termed homotopy equivalence, which is not as strict as homeomorphism.

Def $\mathbb{X}$ and $\mathbb{Y}$ are homotopy equivalent, or have the **same homotopy type**, if there exists continuous maps $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$ such that $f \circ g \simeq id_{\mathbb{Y}}$ and $g \circ f \simeq id_{\mathbb{X}}$.

We denote $\mathbb{X} \simeq \mathbb{Y}$.

Note we have $f \circ g \simeq id_{\mathbb{Y}}$ and $g \circ f \simeq id_{\mathbb{X}}$, and not **equal to** in each case.





notice that $\mathbb{X} \simeq \mathbb{Y}$ here, but $\mathbb{Y}$ is not a retract of $\mathbb{X}$.

If two spaces are homeomorphic, they have the same homotopy type.

So, $\mathbb{X} \approx \mathbb{Y} \implies \mathbb{X} \simeq \mathbb{Y}$.

The implication does not go the other way, as many of the above examples show. For instance, $\mathbb{X}$ (fat 'A') is a 2-manifold with boundary, and $\mathbb{Y}_2$ (1-D 'A'), its 'skeleton', is a 1-manifold with boundary.