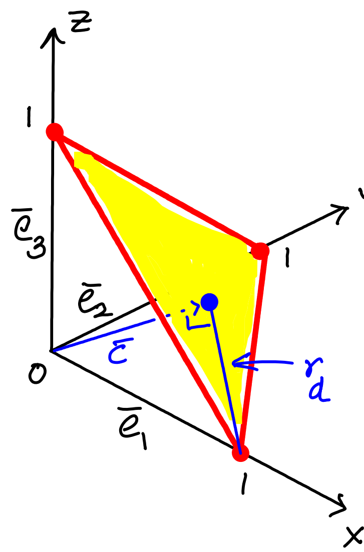
MATH 529: Lecture 12 (02/19/2026)

Today:
- * Voronoi diagram
- * Delaunay triangulation
- * filtration

Recall VR Lemma: $VR_S(r) \subseteq \check{C}ech(\sqrt{2}r)$

Proof idea continued...

Let $\bar{c}$ be the barycenter of Δ^d. → regular d-simplex



$$\bar{c} = \begin{bmatrix} \frac{1}{d+1} \\ \vdots \\ \frac{1}{d+1} \end{bmatrix} \qquad \|\bar{c}\| = \frac{1}{\sqrt{d+1}}$$ is the length from origin of Δ^d.
↳ perpendicular distance

We compute $r_d = \sqrt{\frac{d}{d+1}} \left(= \sqrt{1 - \|\bar{c}\|^2} \right)$.

Note: $r_d \longrightarrow 1$ as $d \to \infty$.

The pairwise distance between $\bar{e}_i$ and $\bar{e}_j$ in σ is $\sqrt{2}$.
Also, the miniball of Δ^d has radius r_d.
Hence, simplex Δ^d of diameter $\sqrt{2}$ also belongs to $\check{C}ech(r_d)$. Multiplying by $\sqrt{2}r$, we get that,

$$VR(r) \subseteq \check{C}ech(\sqrt{2}\, r\, r_d). \qquad \text{But } r_d \leq 1,$$

and hence, $VR(r) \subseteq \check{C}ech(\sqrt{2}r)$. $\square$

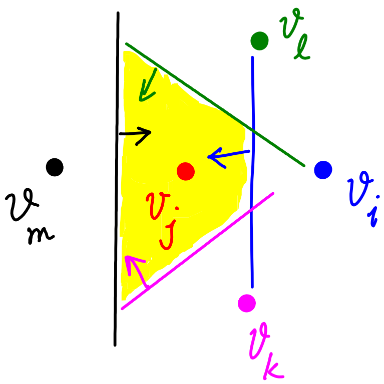
We saw $\check{C}ech_S(r)$ and $VR_S(r)$ of S (points) in $\mathbb{R}^d$.
Can we limit the dimension of the simplices we get from S? Yes!
We can build the Delaunay complex. We first describe its dual construction.

Voronoi Diagram

Recall: $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a finite set of points in $\mathbb{R}^d$.

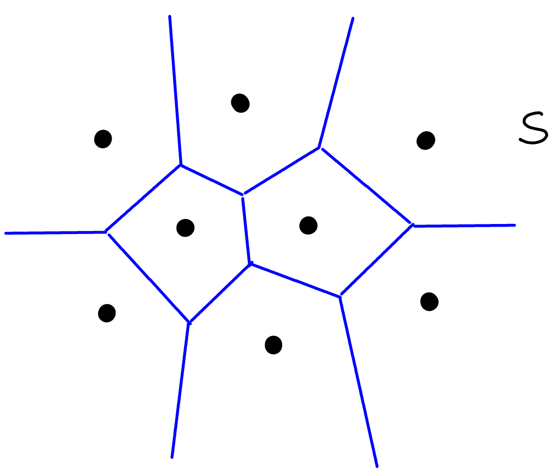
The Voronoi cell of $\bar{v}_j \in S$ is the set of points in $\mathbb{R}^d$ closest to $\bar{v}_j$:

$$V_{\bar{v}_j} = \{\bar{x} \in \mathbb{R}^d \mid \|\bar{x} - \bar{v}_j\| \leq \|\bar{x} - \bar{v}_i\| \ \forall\, \bar{v}_i \in S\}.$$



When we have just two points, say, v_i and v_j, the perpendicular bisector between them is the set of points equidistant from both of them. The half plane on the side of v_j then is V_{v_j}, its Voronoi cell.

$V_{\bar{v}_j}$ is a convex polyhedron, as it is the intersection of a set of half spaces, each being convex. $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ together tile or cover all of $\mathbb{R}^d$.



The collection of $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ is called the **Voronoi diagram** of S.

$V_{\bar{v}_i}$ and $V_{\bar{v}_j}$ meet at most in a common boundary. In $\mathbb{R}^2$, Voronoi cells meet at points or edges.

Notice that $V_{\bar{v}_j}$ can be open or closed. Intuitively, the boundary of $V_{\bar{v}_j}$ can be thought of as the "fence" around $\bar{v}_j$'s "house" — everything within the fence "belongs" to $\bar{v}_j$.

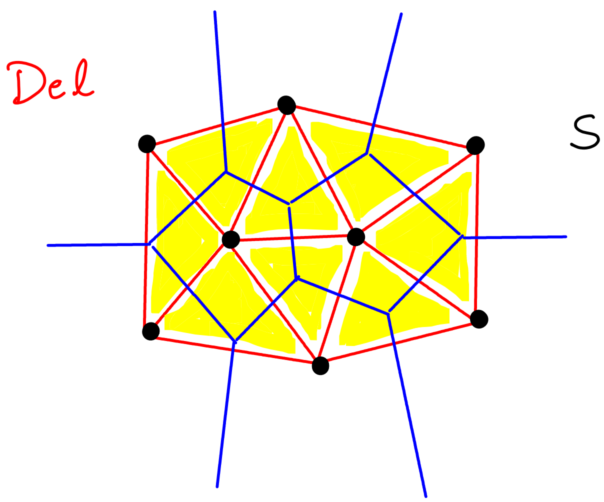
Delaunay Triangulation

The **Delaunay complex** of S is (isomorphic to) the nerve of its Voronoi diagram.

$$\mathrm{Del}_S = \{\sigma \subseteq S \mid \bigcap_{\bar{v}_j \in \sigma} V_{\bar{v}_j} \neq \phi\}.$$

or $\mathrm{Delaunay}_S$

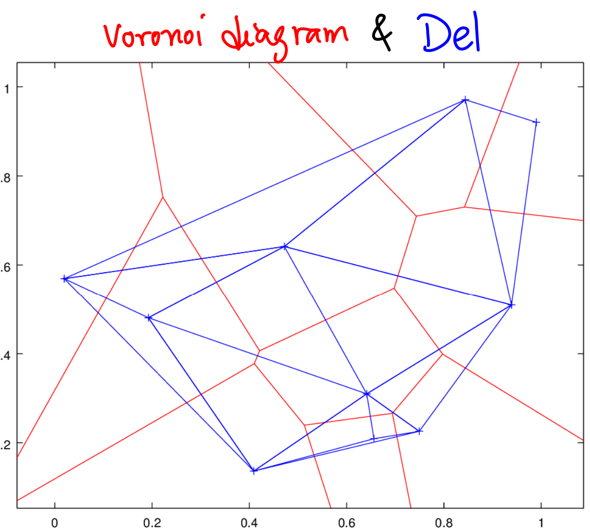
Similar to the Čech complex, we start with a convex set or cell associated with each point in S, and then take the nerve. But instead of balls, we use the Voronoi cells for each vertex.

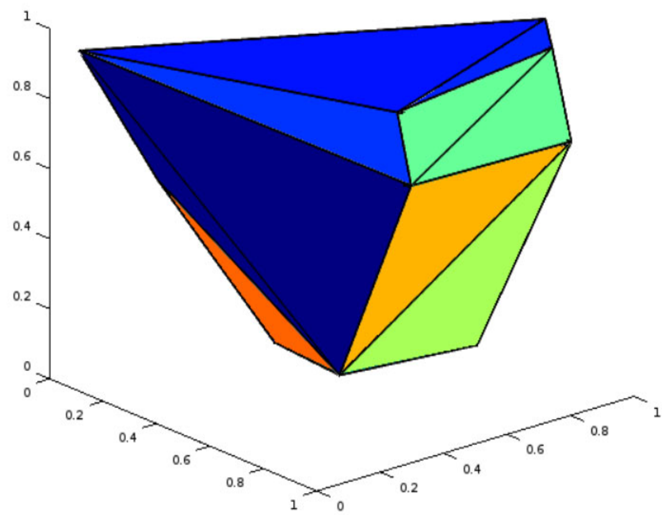


Shown here is one geometric realization of $\mathrm{Delaunay}_S$. This is the "natural" realization, as well.

Check out the commands voronoi, delaunay, delaunayn, and DelaunayTriangulation, as well as related commands in Matlab. Similar commands are available in Python as well.

Here are a couple sample pictures of 2D and 3D Delaunay complexes produced in Matlab. The 2D picture shows the Voronoi diagram as well.

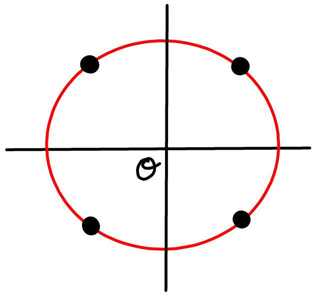




Looks like we only get (upto) triangles here. Recall one of the main motivations we stated for introducing Delaunay complexes — that we wanted to get only upto d-simplices for point sets S in $\mathbb{R}^d$.

Q. Do we always get only triangles in Del_S is 2D?

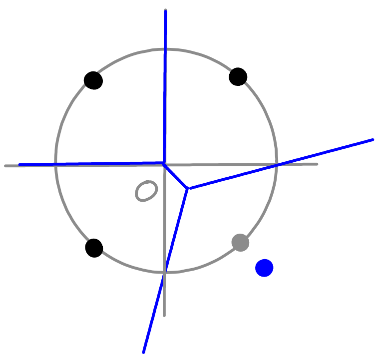
No!



The Voronoi cells V_{v_j} of the four points meet at the central point o (origin) here.

Hence, Del_S contains the tetrahedron!

But, if we move even one of the four points just ever so slightly away from the circle, we can avoid the 4-way intersection of their Voronoi cells.



Mathematically, we need to move only one (out of the four) points by $\varepsilon > 0$ in one of the coordinate directions; ε could be really small, as long as it is > 0.

Def The set of points S in $\mathbb{R}^d$ is in **general position** if no $(d+2)$ points in S lie on a common $(d-1)$-sphere.

e.g., $d = 2 \Rightarrow$ no 4 points lie on a circle.

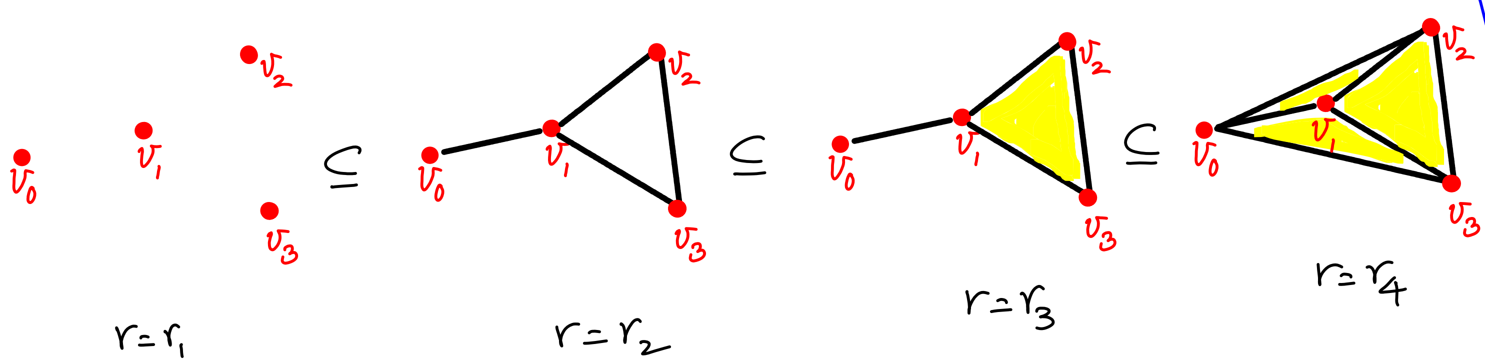
No $(d+2)$ V_{v_j}'s have a common intersection $\Longrightarrow$ ($\sigma \in \mathrm{Del}_S$ means $\dim \sigma \leq d$)

We assume general position usually. If this condition is not satisfied, we could perturb a single coordinate of a single point from any $(d+2)$ such points by a small $\varepsilon > 0$.

There are very efficient (polynomial time) algorithms for constructing Delaunay tessellations, at least in 2D and 3D.

We have seen 3 families of simplicial complexes: $\check{C}ech_S(r)$, $VR_S(r)$, and Del_S. Note that the Delaunay complex is independent of any distance cutoffs (or radii of balls). But, what do we gain by varying r?

Given S, we could study the family of $\check{C}ech_S(r)$ or $VR_S(r)$ as r varies from 0 to ∞ (or, even from $-\infty$ to $+\infty$). Since S is finite, we will only a finite number of such complexes.



$\subseteq$ tetrahedron $v_0v_1v_2v_3$ (at $r=r_5$)

Since there are only four points here, their tetrahedron is the largest dimensional simplex we get in $\check{C}ech(r)$, even if we keep increasing r beyond r_5.

We capture the fact that v_1, v_2, v_3 are closer to each other than, say, $\{v_2, v_3\}$ are to v_0, since $\Delta v_1v_2v_3$ comes in to $\check{C}ech(r)$ before the other triangles.

We will study such families of complexes in detail.

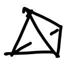
Def A **filtration** of a simplicial complex K is a nested sequence of subcomplexes of K

$$\phi = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = K.$$

The simplicial complex K along with a filtration is a **filtered simplicial complex**.

A **filtration ordering** is a full ordering of all simplices in K such that each prefix of the ordering is a subcomplex.

Filtration ordering: Example

$K =$  $v_0 v_1 v_2 v_3$

a subcomplex

$$v_0 < v_1 < v_2 < v_3 < v_0v_1 < v_0v_2 < v_0v_3 < v_1v_2 \,|\, < v_1v_3 < v_2v_3$$

$$< v_0v_1v_2 < \cdots < v_0v_1v_2v_3.$$

More generally, '$<$' could assign the same rank for several simplices, e.g., all vertices $<$ all edges $<$ all triangles $< \cdots$. In this case, $<$ is not a full ordering but we can convert it to one by breaking ties arbitrarily.

Q. What do we use filtrations for?

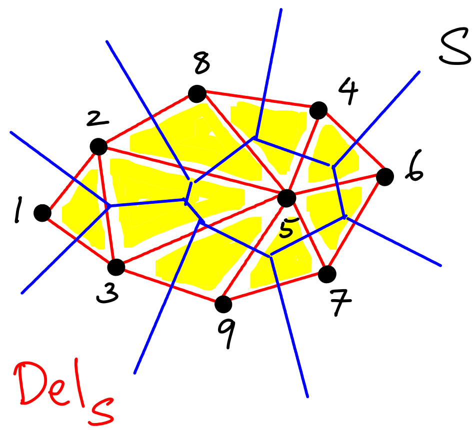
We could study signature functions: $\lambda : \{0, 1, \ldots, m\} \longrightarrow \mathbb{R}^d$, $d \geq 1$. λ assigns a value in $\mathbb{R}^d$ for each $k \in \{0, \ldots, m\}$. We could compare the signatures for two point sets S_1 and S_2 to distinguish them — by comparing $\lambda(S_1)$ and $\lambda(S_2)$.

For instance, χ (Euler characteristic). We could compute $\chi(K^i)$ for each $i \in \{0, \ldots, m\}$, and study the Euler characteristic curve (or vector).

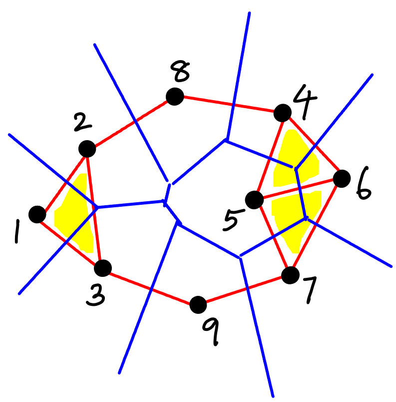
But we will study other more involved signatures soon.

Could we vary r to create a family of nested Delaunay complexes?
Equivalently, could we create a filtration for Del_S?

Consider S with 9 points shown here:



Observe: {1,2,3} and {4,5,6,7} form two clusters of nearby points, which are further away from each other.



A subcomplex of Del_S as shown here would capture the topology of S "better".

For the given set of 9 points, how do we define the complex shown here?
We'll introduce the alpha complex in the next lecture...