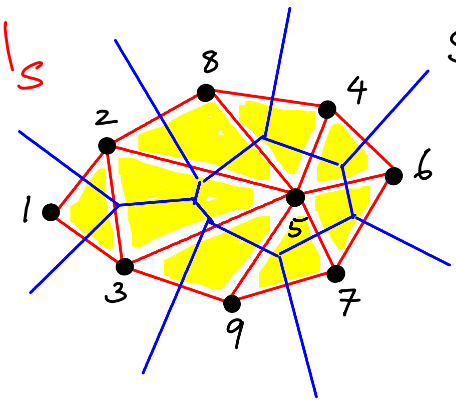
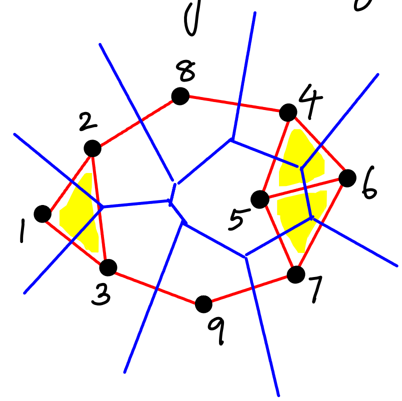
MATH 529: Lecture 13 (02/24/2026)

Today: * alpha complexes
* weighted alpha complexes
* witness complex

Recall A motivating instance of Delaunay triangulation:



Delaunay triangulation of 9 points: {1,2,3} and {4,5,6,7} form clusters farther away from other points.



A subcomplex of Del_S that captures the structure better.

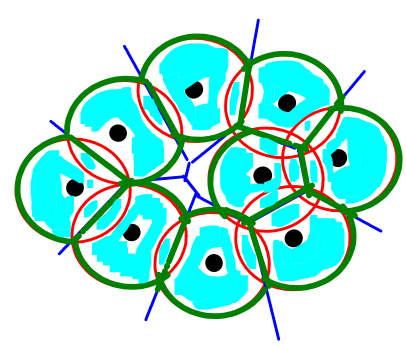
For the given set of 9 points, how do we define the complex shown here?

Alpha Complexes

Let $S = \{\bar{v}_1, \dots, \bar{v}_n\}$, $\bar{v}_j \in \mathbb{R}^d$, $r \geq 0$. Recall that $B_{\bar{v}_j}(r) = \bar{v}_j + r\mathbb{B}^d$
$= \{\bar{x} \in \mathbb{R}^d \mid \|\bar{x} - \bar{v}_j\| \leq r\}$ is the r-ball (d-dimensional) centered at $\bar{v}_j$.

$\bigcup_{\bar{v}_j \in S} B_{\bar{v}_j}(r)$

We "combine" balls around the points, and their Voronoi cells.
We consider $B_{\bar{v}_j}(r) \cap V_{\bar{v}_j}$.



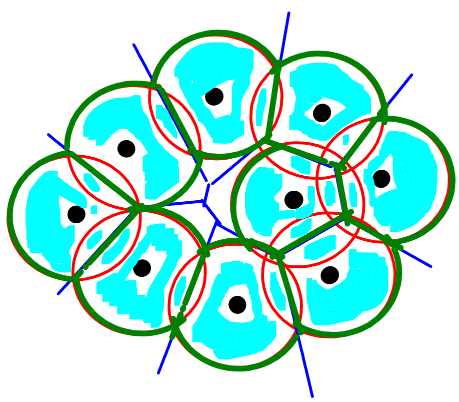
Note that there is a "hole" in the union of regions (in the middle).

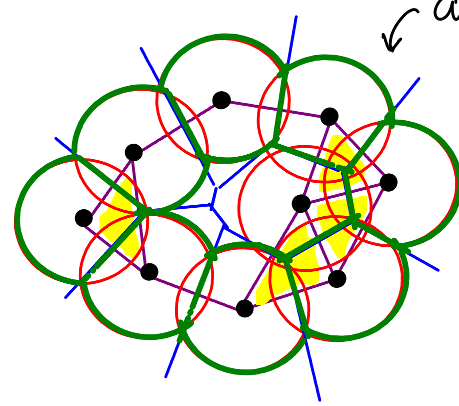
The idea is to combine the good properties of balls and Voronoi cells - get at most d-simplices, but still get a hierarchy.

Let $R_{\bar{v}_j}(r) = B_{\bar{v}_j}(r) \cap V_{\bar{v}_j}$ (region of $\bar{v}_j$). Since both $B_{\bar{v}_j}(r)$ and $V_{\bar{v}_j}$ are convex, so is $R_{\bar{v}_j}(r)$. The regions $R_{\bar{v}_j}(r)$ intersect, if at all, along common boundaries, and together they tile (or, cover) $\bigcup_{\bar{v}_j \in S'} B_{\bar{v}_j}(r)$.

The **alpha complex** is the nerve of this union.

$$\text{Alpha}_S(r) = \{\sigma \subseteq S \mid \bigcap_{\bar{v}_j \in \sigma} R_{\bar{v}_j}(r) \neq \phi\}.$$





$R_{\bar{v}_j}(r) \subseteq V_{\bar{v}_j} \implies \text{Alpha}(r) \subseteq \text{Del}_S$. Also,

$R_{\bar{v}_j}(r) \subseteq B_{\bar{v}_j}(r) \implies \text{Alpha}(r) \subseteq \check{C}ech_S(r)$.

By the nerve lemma, $\bigcup_{\bar{v}_j \in S'} B_{\bar{v}_j}(r) \simeq |\text{Alpha}(r)|$.

By varying r and considering $\text{Alpha}(r)\ \forall r$, we get a filtration of the Delaunay complex.

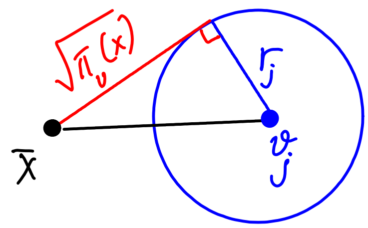
Q: Can we use balls of different radii?

Motivation: modeling proteins – made up of atoms, which could be modeled using balls of different radii.

Weighted Alpha Complexes

Def The weighted squared distance, or the **power distance**, of $\bar{x} \in \mathbb{R}^d$ from $\bar{v}_j$ with weight w_j is

$$\pi_{\bar{v}_j}(\bar{x}) = \|\bar{x} - \bar{v}_j\|^2 - w_j.$$



When $w_j = r_j^2$, we get $\pi_{\bar{v}_j}(\bar{x})$ is the squared length of the tangent from $\bar{x}$ to $B_{\bar{v}_j}(r_j)$.

The **weighted** or **power Voronoi** cell of $\bar{v}_j$ is

$$W_{\bar{v}_j} = \{ \bar{x} \in \mathbb{R}^d \mid \pi_{\bar{v}_j}(\bar{x}) \leq \pi_{v_i}(\bar{x}) \; \forall \bar{v}_i \in S \}.$$

weighted Voronoi diagram

The **power Voronoi complex** is the collection of $W_{\bar{v}_j}$'s. And the **weighted Delaunay complex** is the nerve of the **weighted Voronoi diagram**.

We could apply the concept of weighted voronoi diagram also to sets of points where different points have different weights (and not only to cases where the balls are differently sized).

The definition of power distance appears somewhat involved – we will describe the motivation soon!

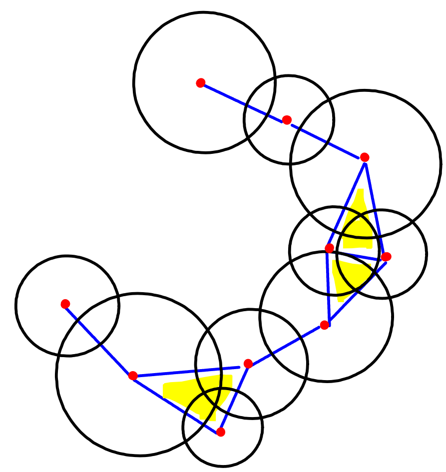
Weighted Alpha Complex

We set $w_j = r_j^2$ and define

$$R^w_{\bar{v}_j}(r) = B_{\bar{v}_j}(r) \cap W_{\bar{v}_j}$$

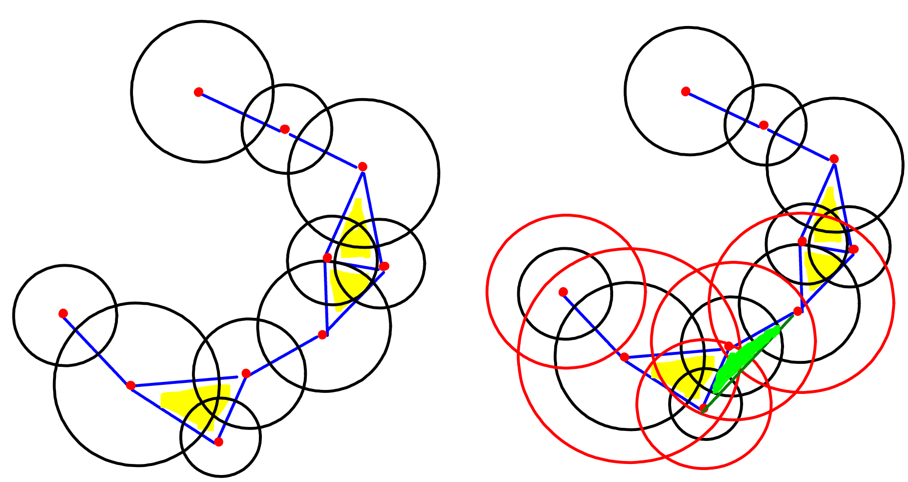
The **weighted alpha complex** is the nerve of the collection of $R^w_{\bar{v}_j}(r)$. This complex is a subcomplex of the weighted Delaunay complex.

Here is an illustration. The discs could model different atoms, and the weighted alpha complex shown here is one "skeleton" of the protein, which is the collection of atoms.



Here is an illustration of how the weighted alpha complex grows as we increase r_j^2 linearly, i.e., we set $w_j = r_j^2 \leftarrow r_j^2 + r$ and let $r \to \infty$.

A subset of the bigger balls (2D-discs in this case) are shown here for illustration. The extra triangle added to the original nerve is shown in green.



We mentioned varying $r_j^2 \leftarrow r_j^2 + r$. Why do it that way?

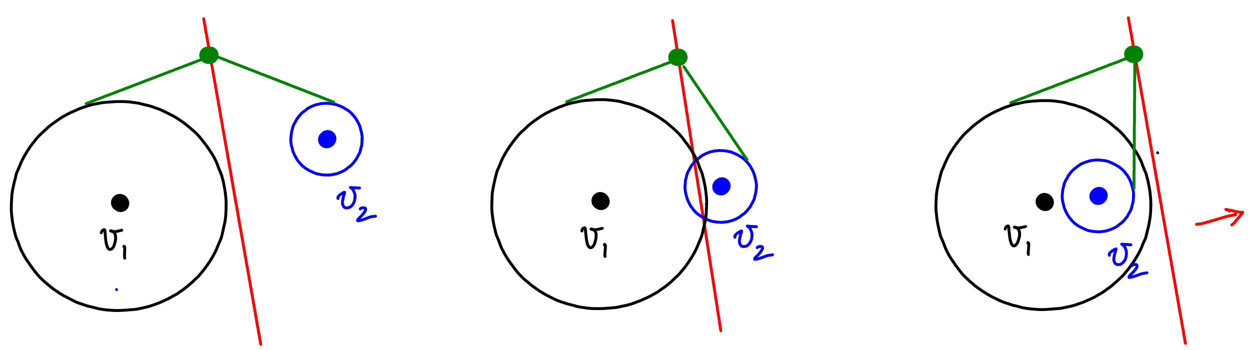
In general, starting with the $R^w_{\bar{v}_j}$ cells, we could "vary" the different radii to get a weighted alpha complex filtration

$\phi = K^0 \subseteq \cdots \subseteq K^m =$ weighted Delaunay complex.

Q: How to "vary" the radii? r_j are not same to start with.

1. Set $w_j = r_j^2$, and then increase all radii r_j at the same linear rate, i.e., $r_j \leftarrow r_j + r$ for $r > 0$. Then let $r \to \infty$ (uniformly increase all r_j).
   But, $W_{\bar{v}_j}$ for different r may not be the same. Hence, it could happen that $K^j \nsubseteq K^{j+1}$ for some j. This situation is best avoided!

Here is an observation: Bisector of two weighted points



Note that the bisector stays a straight line!

Similar to the default alpha complex construction, where the Voronoi cells stay the same while the balls grow, it is desirable to have the weighted Voronoi cells stay same as well.

2. We set $w_j = r_j^2$, and grow the square of the radii uniformly, i.e., set $r_j^2 \leftarrow r_j^2 + r$, as $r \to \infty$.

Since we are using the power distance,

$$\pi_{\bar{v}_j}(\bar{x}) = \pi_{\bar{v}_i}(\bar{x}) \Rightarrow \|\bar{x} - \bar{v}_j\|^2 - (r_j^2 + \cancel{r}) = \|\bar{x} - \bar{v}_i\|^2 - (r_i^2 + \cancel{r})$$

Hence, the bisectors using $\pi_{\bar{v}}(\bar{x})$ stay the same as $r \to \infty$.

So, the power Voronoi cells remain the same, just like $V_{\bar{v}_j}$.

So, as r increases, $W_{\bar{v}_j}$ remains same. We do get the nesting of simplicial complexes as r increases.

In fact, $W_{\bar{v}_j}$'s here have most of the nice properties that $V_{\bar{v}_j}$, the default Voronoi cells, have. As such, the alpha complex filtration also has most nice properties that the default (same r for each r_j) alpha complexes.

Originally, Edelsbrunner and Mücke (1983) defined the weighted alpha complexes to study structure of biomolecules. The notation used was $r_j^2 \leftarrow r_j^2 + \alpha$ for the growth parameter α $(-\infty < \alpha < \infty)$. Hence the name alpha complex.

There are efficient algorithms to construct the weighted alpha complexes in 2D and 3D. We will discuss a version in a future lecture.

For large sets of points, all of the complexes we introduced — Čech, VR, alpha, etc., become intensive to compute. We would be better off sampling a subset of points!

Čech and VR complexes grow too large even for moderately large point sets S. For instance, the VR complex of a set with ~2000 points in $\mathbb{R}^3$ could have more than a million triangles! Further, computing Delaunay and alpha complexes are also computationally expensive in high dimensions. We look at a possible alternative now.

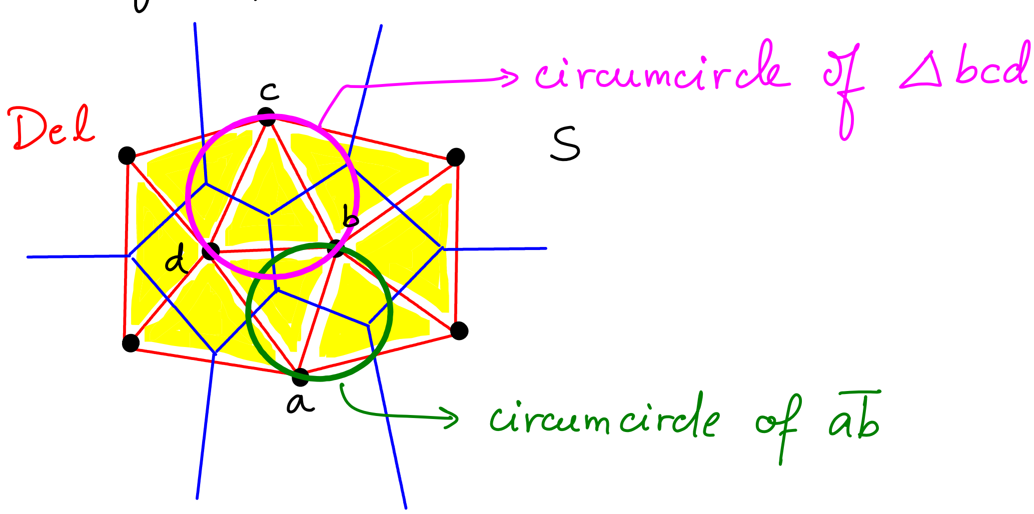
We will now consider two families of complexes that are designed to be much more sparse — they sample from the input point set, and build the complex by generalizing or relaxing some of the conditions used to define the complexes we have already seen.

A crucial property of the Delaunay complex

The **empty circumsphere** property: boundary of miniball

$\sigma \in \mathrm{Del}_S \iff$ circumsphere(σ) has no points of S in its interior.

Of course, with $\sigma = \operatorname{conv}\{\bar{v}_0, \dots, \bar{v}_k\}$, $\bar{v}_0, \dots, \bar{v}_k$ lie on the circumsphere, and the center of its circumsphere is in the intersection of the Voronoi cells of $\bar{v}_0, \dots, \bar{v}_k$.



Witness Complex

Idea: Choose $L \subseteq S$ (L is typically small), and build the complex only on L. Use the remaining points in $S \setminus L$ as possible "witnesses" for the simplices in the complex.

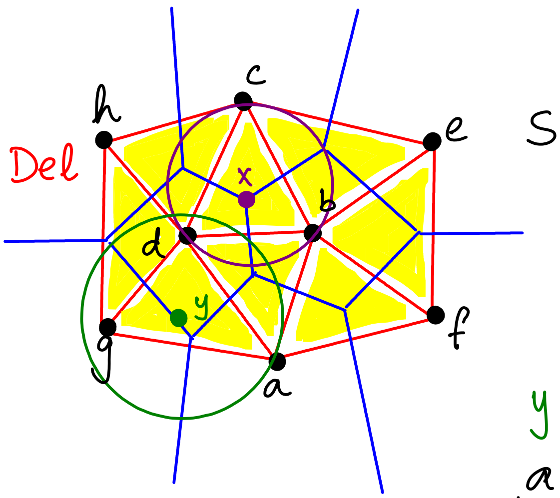
Def Let $\sigma = [\bar{v}_0 \dots \bar{v}_k]$, $\bar{v}_j \in S$, be a k-simplex, where $\bar{v}_j \in \mathbb{R}^d$. $\bar{x} \in \mathbb{R}^d$ is a **weak witness** for σ with respect to S if $\|\bar{x} - \bar{v}\| \leq \|\bar{x} - \bar{u}\|$ $\forall$ $\bar{v} \in \{\bar{v}_0, \dots, \bar{v}_k\}$ and $\bar{u} \in S \setminus \{\bar{v}_0, \dots, \bar{v}_k\}$.

$\bar{x} \in \mathbb{R}^d$ is a **strong witness** for σ w.r.t. S if it is a weak witness, and in addition, $\|\bar{x} - \bar{v}_0\| = \|\bar{x} - \bar{v}_1\| = \dots = \|\bar{x} - \bar{v}_k\|$.

Equivalently, we say that σ is weakly (or strongly) witnessed by $\bar{x}$.

We could define the Delaunay complex in terms of existence of strong and weak witnesses for the simplices. Subsequently, we could relax (some of) the requirements to build complexes that are more manageable in size.

We first illustrate weak and strong witness points on the Delaunay complex we have seen previously.



x is a strong witness for $\triangle bcd$.
Notice that x is the center of the circumcircle of $\triangle bcd$, which is also the (point of) intersection of the Voronoi cells of the vertices b, c, and d.

y is a weak witness for $\overline{dg}$, and also for $\triangle adg$. Notice that as drawn, $\|y-g\| < \|y-d\| < \|y-a\| < \|y-v\|$ for $v = b, c, e, f, h$.