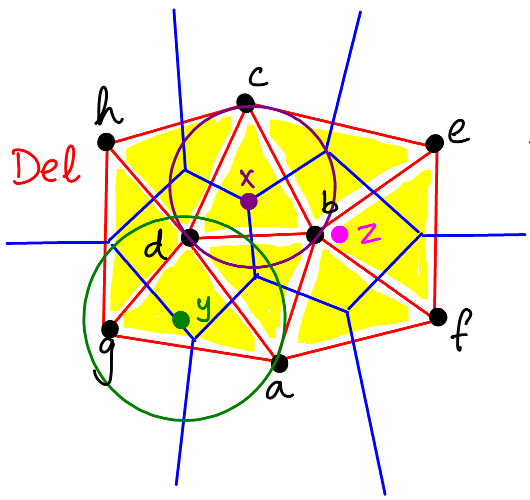
MATH 529: Lecture 14 (02/26/2026)

Today: * witness complex
* background on groups

Recall: weak and strong witness of $\sigma = [v_0 \dots v_k]$ in S...

Continuing with the illustration from Lecture 13...



z is a weak witness for Δabe. Also, Δabe as shown does not have a strong witness. Intuitively, the center of the circumcircle of Δabe lies closer to f than to a, b, and e.

Indeed, every simplex in Del_S will have a strong witness, which is the center of the empty circumsphere of that simplex. Ideally, we want to sample from S, and look for witnesses for simplices in the points not included in the sample. At the same time, looking for a strong witness among a given set of vertices ($\subseteq S$) is futile, as such a point occurs with zero probability. But the following result bails us out.

Result (de Silva, 2003) $\sigma = [v_0 \dots v_k] \subseteq S$ has a strong witness iff every $\tau \leq \sigma$ (face) has a weak witness.

Notice that one direction is obvious — if σ has a strong witness, the same point is a strong witness for all its faces too, and hence every face has a weak witness. The other direction is more technical — see the paper posted on the course web page for details.

We define witness complexes in the more general setting of a metric space with pairwise distances between points provided.

Let $D = [d_{ij}]$ be the $l \times n$ distance matrix between $L \subseteq S$ of landmark points and all points in S. Here, $|L| = l$, and $|S| = n$.

Def The (strict) **witness complex** $W_\infty(L,S)$ is the collection of all simplices $\sigma \subseteq L$ whose all subsimplices have weak witnesses in S.

This restriction of all faces having weak witnesses is required to insure that $W_\infty(L,S)$ is a simplicial complex. Notice that we are building simplices on points just from L, and not from all of S.

In particular, if $\sigma = [v_0 \cdots v_k] \in W_\infty(L,S)$, then there exists a j with $1 \le j \le n$ such that d_{ij} for $i = v_0, \dots, v_k$ are the $(k+1)$ smallest entries in the j^{th} column of D in some order.

We also say that $\bar{u}_j \in S$ (corresponding to the j^{th} column) is a witness to the existence of σ in $W_\infty(L,S)$. s.t.

Relationship to Del_L

Result $\sigma \in Del_L$ iff σ is strongly witnessed.
(just follows from the empty circumsphere property).

But $\sigma \in W_\infty(L,S)$ implies that σ is strongly witnessed, as all its faces have weak witnesses. Hence, we get that

$$W_\infty(L,S) \subseteq Del_L.$$

So, we are first choosing a possibly much smaller set L of points from S as landmarks. We then build a complex on L which also has a bound on the dimension of the simplices being included.

Similar to how we defined the Vietoris-Rips complex by relaxing the definition of the Čech complex, we now define an easier to construct version of the strict witness complex by requiring that only edges need to be present for a higher dimensional simplex to be included.

Def The **lazy witness complex** $W_1(L,S) \supseteq W_\infty(L,S)$ has the same 1-skeleton as $W_\infty(L,S)$. After that, $\sigma = [v_0 \cdots v_k] \subseteq L$ is in $W_1(L,S)$ iff all edges of σ are in $W_1(L,S)$.

In practice, we almost always work with the lazy witness complex $W_1(L,S)$, and write $W(L,S)$ to mean $W_1(L,S)$ (and not $W_\infty(L,S)$).

Q: Is $W_1(L,S) \subseteq \mathrm{Del}_L$? If not, can you give a counterexample?

Think about it!

How to choose the landmarks L

First decide how many landmarks you want ($|L| = \ell$). Then,

two methods
- random selection (select ℓ points randomly)
- maxmin selection.

Maxmin selection of ℓ landmarks

Choose the first landmark ℓ_1 randomly. After that, inductively, with $\{\ell_1, \dots \ell_{i-1}\}$ chosen, pick ℓ_i that maximizes the following function:

$$z \mapsto \min\{D(z,\ell_1), D(z,\ell_2), \dots, D(z,\ell_{i-1})\},$$ where

$D(z,\ell_j)$ is the distance from z to ℓ_j.

Maxmin provides widespread coverage of S, but could also end up picking outliers.

Guidelines for choosing $\ell = |L|$

(de Silva, Carlsson, 2004) for data sampled from surfaces (in 3D), $\ell \leq \frac{n}{20}$ works reasonably well.

Javaplex and Gudhi provide functions to build witness complexes.

https://github.com/appliedtopology and https://gudhi.inria.fr/.

Now that we have seen how to build several simplicial complexes on point sets, we will talk about how to infer the topology of the built complex. We first review basic results from algebra, and use them to define and study groups on the simplicial complex.

Review of Algebra (Groups)

A binary operation $*$ on a set S is a rule that assigns to each ordered pair $(a,b) \in S$ some element in S.

e.g., $a*b=c$, for $c \in S$.

If $a*b=b*a$ $\forall a,b \in S$, $*$ is **commutative**.

$(a*b)*c = a*(b*c)$ $\forall a,b,c \in S \Rightarrow *$ is **associative**.

A **group** $\langle G, * \rangle$ is a set G with a binary operation $*$ defined on elements of G such that the following conditions hold.

(a) $*$ is associative.

(b) $\exists e \in G$ such that $e*a=a*e=a$ $\forall a \in G$.

e is an identity element for $*$ on G.

(c) $\forall a \in G$, $\exists a' \in G$ such that $a*a'=a'*a=e$.

a' is the inverse of a with respect to $*$.

We assume that G is **closed under** $*$ to begin with, i.e., $\forall a,b \in G$, $a*b=c$ for some $c \in G$.

If G is finite, the **order** of the group is $|G|$. Oftentimes, G itself is used to denote the group, with operation $*$ understood.

e.g., $\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{Z}_4, +_4 \rangle$

integers

$\{0,1,2,3\}$

add modulo 4

$\mathbb{Z}_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

If $*$ is commutative, G is an **Abelian group**. We also say that G is **abelian**.

$\mathbb{Z}$ and $\mathbb{Z}_4$ are both abelian. Also, notice that the order of $\mathbb{Z}_4$ is 4.

__Induced operation__ Let $\langle G, *\rangle$ be a group, and $S \subseteq G$. If S is closed under $*$, then $*$ is the **induced operation** on S from G.

__Subgroup__ $H \subseteq G$ (subset) is a **subgroup** of $\langle G, *\rangle$ if H is a group and is closed under $*$.

$\{e\}$ is the trivial subgroup of G. All other subgroups are nontrivial.

Theorem $H \subseteq G$ of a group $\langle G, *\rangle$ is a subgroup of G if and only if

(a) H is closed under $*$;

(b) identity e of G is in H; and

(c) $\forall a \in H$, $a^{-1} \in H$.

This theorem could also be used as the definition of a subgroup.

e.g., Consider $\mathbb{Z}_4$

$H = \{0, 2\}$ is a subgroup.

Notice that $2+2 = 0 \pmod 4$, and hence H is indeed closed under the operation in question (addition modulo 4).

$\mathbb{Z}_4$	$\boxed{0}$	1	$\boxed{2}$	3
$\boxed{0}$	$\boxed{0}$	1	2	3
1	1	2	3	0
$\boxed{2}$	$\boxed{2}$	3	$\boxed{0}$	1
3	3	0	1	2

$H = \{0, 2\}$ is the only nontrivial subgroup of $\mathbb{Z}_4$ (distinct from $\mathbb{Z}_4$ itself).

$a^{-1} * b$

<u>Cosets</u> Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by $a \sim_L b$ iff $a^{-1}b \in H$. Similarly, $\sim_R$ is defined on G by $a \sim_R b$ iff $ab^{-1} \in H$. Note that $\sim_L$ and $\sim_R$ are equivalence relations on G. Also

$$a^{-1}b \in H \text{ or } a^{-1}b = h \in H \Longrightarrow b = ah.$$

For $a \in G$, the subset $aH = \{ah \mid h \in H\}$ of G is the **left coset** of H containing a, and $Ha = \{ha \mid h \in H\}$ is the **right coset** of H containing a.

If G is abelian, then $ah = ha$ $\forall a \in G, h \in H$. Then the left and right cosets match, i.e., $aH = Ha$.

e.g., $\mathbb{Z}_4$, $H = \{0, 2\}$ is a subgroup. The coset of 1 is $1 +_4 \{0,2\} = \{1,3\}$.

Our goal is to use groups to characterize topological spaces. Hence, we need to be able to characterize the "structure" of groups. We could use maps between groups for this purpose.

To simplify notation, we write ab for $a * b$, with $*$ understood.

<u>Homomorphisms</u> A map $\varphi : G \to G'$ is a ***homomorphism*** if $\varphi(ab) = \varphi(a)\varphi(b)$ $\forall a, b \in G$.

$$\varphi(a *_G b) = \varphi(a) *_{G'} \varphi(b)$$

We can always define a trivial homomorphism by setting $\varphi(g) = e'$ $\forall g \in G$, where e' is the identity of G'.

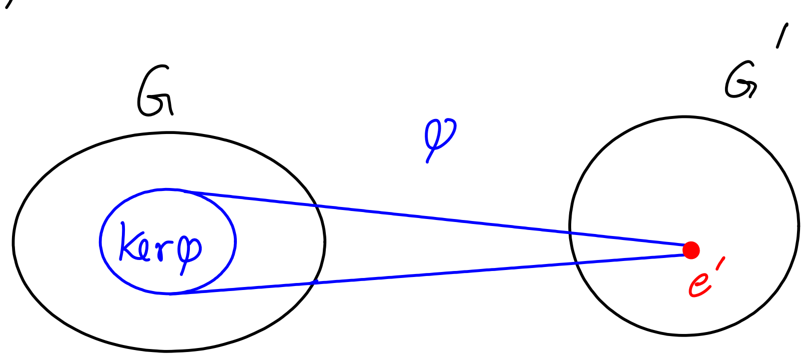
Homomorphisms preserve identity, inverse, and subgroups.

Theorem Let $\varphi: G \to G'$ be a homomorphism. Then

1. $\varphi(e) = e'$, where e, e' are identities of G, G', respectively.
2. $\varphi(a^{-1}) = (\varphi(a))^{-1}$ $\forall a \in G$.
3. $H \subseteq G$ is a subgroup of $G \Rightarrow \varphi(H)$ is a subgroup of G'.
4. $K' \subseteq G'$ is a subgroup of $G' \Rightarrow \varphi^{-1}(K')$ is a subgroup of G.

This theorem could be used, alternatively, as a definition of homomorphism.

Kernel Let $\varphi: G \to G'$ be a homom. The subgroup $\varphi^{-1}(\{e'\}) \subseteq G$ is the **kernel** of φ.



Notice that $\{e'\}$ is the trivial subgroup of G'. Hence by (4) of the Theorem above, $\ker\varphi$ is a subgroup of G.

Since $\ker\varphi$ is a subgroup of G, we can define kernel cosets. Let $H = \ker\varphi$, $a \in G$, then

$$aH = \varphi^{-1}\{\varphi(a)\} = \{x \in G \mid \varphi(x) = \varphi(a)\} = Ha.$$

Intuitively, any $h \in \ker\varphi$ gets mapped to the identity (e'). So, we could add h to a to get x, and x also gets mapped to the image of a.

If $gH = Hg$ $\forall g \in G$ for a subgroup H of G, then H is a **normal** subgroup of G. All subgroups of Abelian groups are normal, and so is $\ker\varphi$ for a homomorphism φ.

The properties of a function being injective, surjective, or both can be studied for homomorphisms as well. But for homomorphisms, we refer to these properties using terms specific to groups.

Maps in general	homomorphisms between groups
1-1	monomorphism
onto	epimorphism
1-1 and onto (bijection)	isomorphism ($\cong$) notation

Isomorphism between groups is like homeomorphism between topological spaces. Recall previous discussion about ASCs being isomorphic.

<u>Finitely generated</u> Let $a_i \in G$ for $i \in I$, an index set. The smallest subgroup of G containing $\{a_i \mid i \in I\}$ is the subgroup generated by $\{a_i \mid i \in I\}$. If this subgroup is all of G, then $\{a_i \mid i \in I\}$ **generates** G, and a_i are the generators. If I is finite, then G is **finitely generated**.

For instance, both $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}_4, +_4 \rangle$ are finitely generated.