MATH 529: Lecture 15 (03/03/2026)

Today: * p-chains, boundary
* p-cycles

Homology Groups

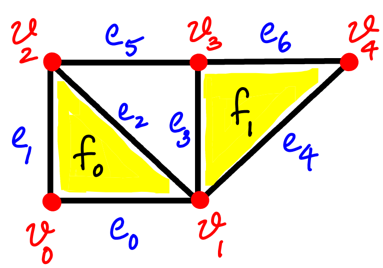
We will talk about homology for simplicial complexes, i.e., simplicial homology. Homology studies "holes" in spaces, i.e., holes in 1D, enclosed voids in 2D, etc. Interestingly, holes are characterized by what surrounds them!

Let K be a simplicial complex. A **p-chain** (for $p \le \dim K$) of K is a formal sum of the p-simplices of K. → linear combination

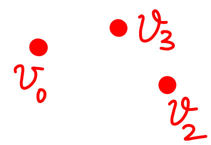
$$\bar{c} = \sum_{i=1}^{m} a_i \sigma_i,$$ where K has m p-simplices, $\sigma_1, \dots, \sigma_m$, and a_i's are their coefficients.

To define groups using addition modulo 2, i.e., in $\mathbb{Z}_2$, $a_i \in \{0,1\}$. When using addition over $\mathbb{Z}$, $a_i \in \mathbb{Z}$. We could define homology groups over $\mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ (→ rational numbers), or over any ring.

Examples

K: 

0-chain (over $\mathbb{Z}_2$)

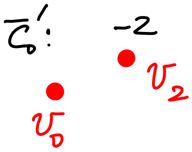
$\bar{c}_0$: 

$a_i = 1$ $i = 0,2,3$

$a_i = 0$, $i = 1,4$

$$\bar{c}_0 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

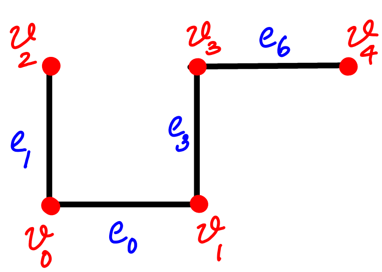
0-chain (over $\mathbb{Z}$)

$\bar{c}_0'$: 

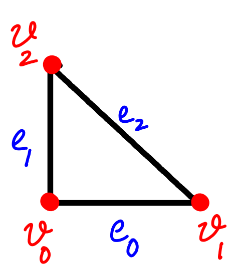
$$\bar{c}_0' = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

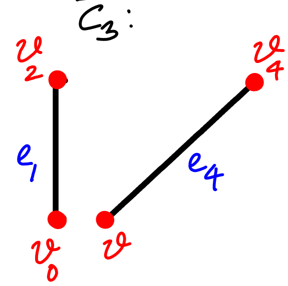
1-chains

over $\mathbb{Z}_2$:

$\bar{c}_1$: 

$$\bar{c}_1 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

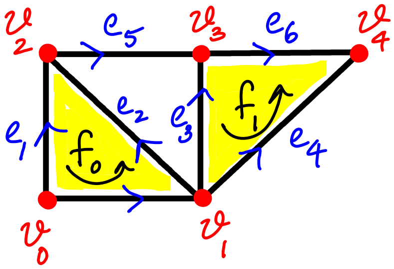
$\bar{c}_2$: 

$\bar{c}_3$: 

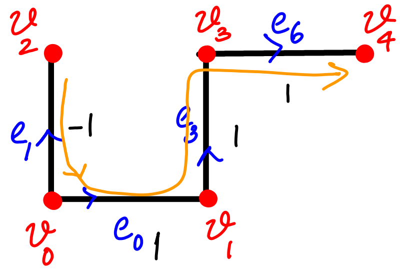
Note that the chain need not be connected.

We consider orientations over $\mathbb{Z}$.

K:



$\bar{c}_1'$:

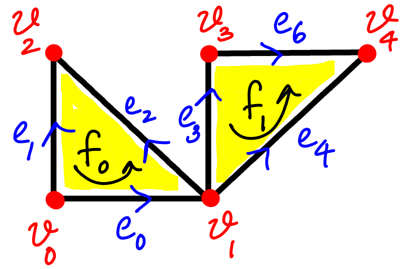


$$\bar{c}_1' = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 1\\-1\\0\\1\\0\\0\\1 \end{bmatrix}$$

we could have an "overall orientation" for the 1-chain, but this is not needed always.

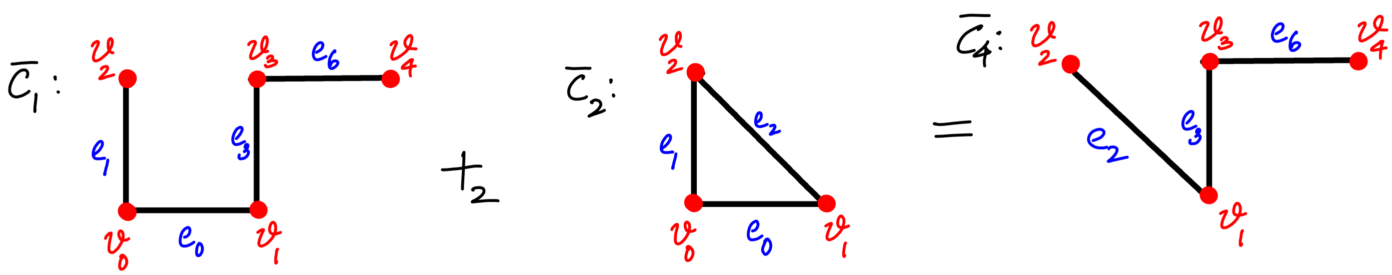
2- chains

$\bar{d}_1 = \begin{array}{c} 0\\1 \end{array}\begin{bmatrix} 1\\1 \end{bmatrix}$ is a 2-chain (over $\mathbb{Z}_2$ or $\mathbb{Z}$)



$\bar{d}_2 = \begin{bmatrix} 2\\-3 \end{bmatrix}$ is a 2-chain over $\mathbb{Z}$.

We can add two p-chains component- or p-simplex-wise, i.e., by adding their vectors. If $\bar{c} = \sum_{i=1}^{m} a_i\sigma_i$, $\bar{c}' = \sum_{i=1}^{m} b_i\sigma_i$, then $\bar{c} +_2 \bar{c}' = \sum_{i=1}^{m} (a_i +_2 b_i)\sigma_i$



Equivalently, we add the corresponding vectors:

$$\underbrace{\begin{bmatrix} 1\\1\\0\\1\\0\\0\\1 \end{bmatrix}\begin{array}{c} e_0\\e_1\\e_2\\e_3\\e_4\\e_5\\e_6 \end{array}}_{\bar{c}_1} +_2 \underbrace{\begin{bmatrix} 1\\1\\1\\0\\0\\0\\0 \end{bmatrix}\begin{array}{c} e_0\\e_1\\e_2\\e_3\\e_4\\e_5\\e_6 \end{array}}_{\bar{c}_2} = \underbrace{\begin{bmatrix} 0\\0\\1\\1\\0\\0\\1 \end{bmatrix}\begin{array}{c} e_0\\e_1\\e_2\\e_3\\e_4\\e_5\\e_6 \end{array}}_{\bar{c}_4}$$

p-chains with addition ($+_2$ over $\mathbb{Z}_2$ or $+$ over $\mathbb{Z}$) form the group of p-chains of K, denoted $\langle C_p(K), +_2\rangle$ or $\langle C_p(K), \mathbb{Z}\rangle$, or simply $C_p(K)$ (or just C_p).

$C_p(K)$ is indeed a group, and is an Abelian group.

* identity: $\bar{0} = \sum_{i=1}^{m} 0\sigma_i$
* inverse: inverse of $\bar{c}$ is $-\bar{c}$ over $\mathbb{Z}$, and $\bar{c}$ over $\mathbb{Z}_2$ (as $\bar{c} +_2 \bar{c} = \bar{0}$).
* $+_2$ and $+$ are associative
* $+_2$ and $+$ are commutative.

For K, there is $C_p(K)$ for $0 \le p \le \dim(K)$.

If K has m p-simplices, each p-chain can be represented by an m-vector. The p-chains corresponding to the m unit vectors are the **elementary p-chains** of K, i.e., they correspond to each σ_i.

$$\bar{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to \sigma_i$$

When K is finite, these m elementary chains generate $C_p(K)$, making it finitely generated.

How are $C_p(K)$ related for various p?

We use "boundary", defined as homomorphisms between $C_p(K)$ and $C_{p-1}(K)$ to connect these chain groups.

We first define the boundary of a single simplex, i.e., an elementary chain, and then extend it naturally to chains. We provide the definition over both $\mathbb{Z}_2$ and $\mathbb{Z}$.

Boundary The **boundary** of a p-simplex is the sum of all its $(p-1)$-faces.

same notation used for orientation, which we do not use over $\mathbb{Z}_2$.

If $\sigma = \operatorname{conv}\{v_0, \dots, v_p\}$, or $\sigma = [v_0 v_1 \dots v_p]$, then

p-boundary

$$\partial_p \sigma = \sum_{j=0}^{p} [v_0 \dots \hat{v}_j \dots v_p],$$ where $\hat{v}_j$ means v_j is omitted.

Over $\mathbb{Z}$, we have $\partial_p \sigma = \sum_{j=0}^{p} (-1)^j [v_0, \dots, \hat{v}_j, \dots v_p]$, which is the sum of all its $(p-1)$-faces with their induced orientations.

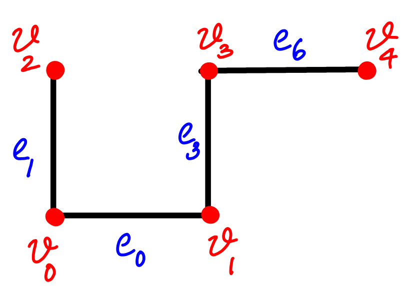
Notice that $\partial_p \sigma$ is a $(p-1)$-chain (in both cases).

The boundary of a p-chain is the sum of the boundaries of its p-simplices.

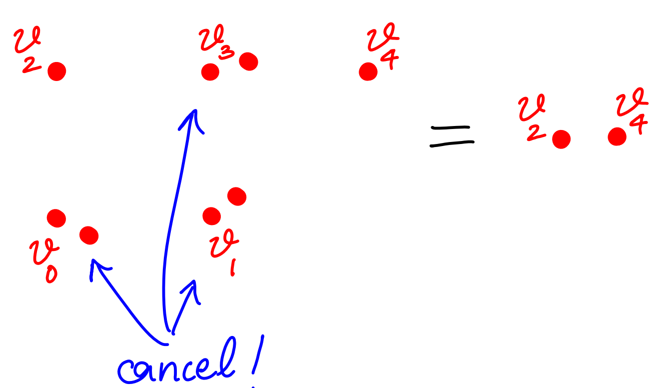
$$\bar{c} = \sum_{i=1}^{m} a_i \sigma_i \Rightarrow \partial_p \bar{c} = \sum_{i=1}^{m} a_i (\partial_p \sigma_i),$$ which is also a $(p-1)$-chain.

Examples

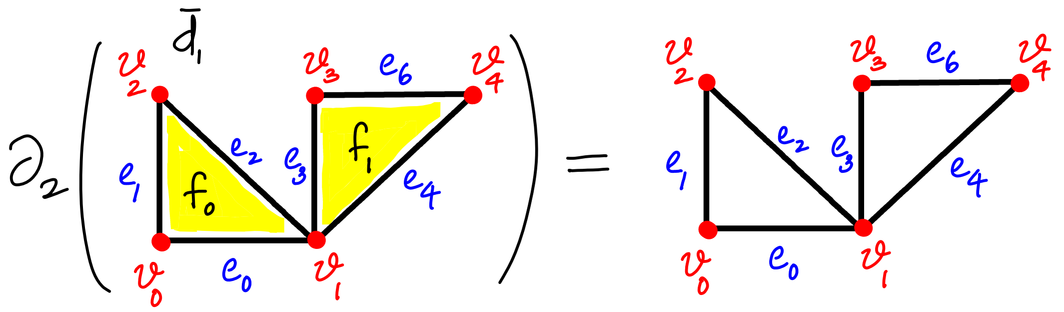
$\bar{c}_1$:



$\Rightarrow \partial_1(\bar{c}_1) =$

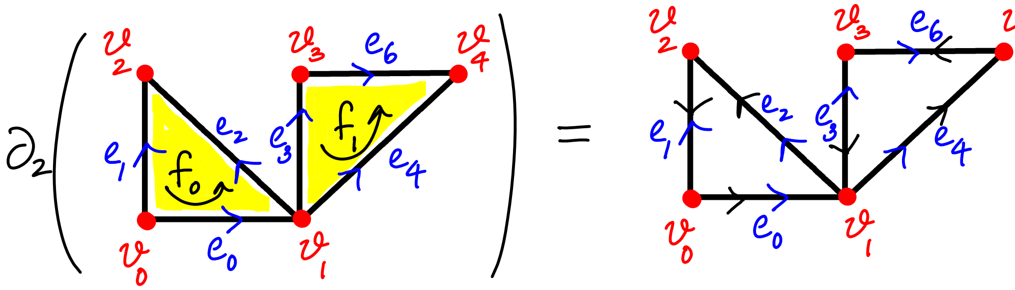


Note that vertices shared by two edges in the chain do not appear in its boundary.



The boundary of a triangle is made of its edges.

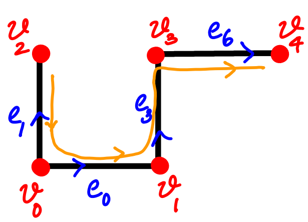
Consider $\partial_2 \bar{d}_1$ over $\mathbb{Z}$ now:



$$\partial_2 \bar{d}_1 = \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Notice that the induced orientation on $\bar{e}_1$ is opposite to its own orientation.

$\bar{c}_1$:



$$\partial \bar{c}_1 = -(v_2 - \cancel{v_0}) + (\cancel{v_1} - \cancel{v_0}) + (\cancel{v_3} - \cancel{v_1}) + (v_4 - \cancel{v_3}) = v_4 - v_2.$$

The dimension is often omitted, and we just talk about $\partial \bar{c}$ of a p-chain $\bar{c}$, with the dimension understood.

Taking the boundary maps a p-chain to a $(p-1)$-chain. Equivalently, we can talk about the map $\partial_p : C_p \to C_{p-1}$. Notice that such a map is defined for each p in the range $1 \le p \le \dim K$.

Also, $\partial_p(\bar{c} +_2 \bar{c}') = \partial_p \bar{c} +_2 \partial_p \bar{c}'$ for 2 p-chains $\bar{c}, \bar{c}'$. Hence ∂_p is a homomorphism, referred to as the **p-th boundary map** or homomorphism.

commutes with $+_2$, which is the definition of a homomorphism

Similarly, we have ∂_{p-1}, which is the $(p-1)$-st boundary homomorphism, and ∂_{p-2}, and so on. $\partial_{p-1} : C_{p-1} \to C_{p-2}$.

∂_p is a homomorphism over $\mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}$) as well, not just over $\mathbb{Z}_2$.

A **chain complex** is a sequence of chain groups connected by boundary homomorphisms

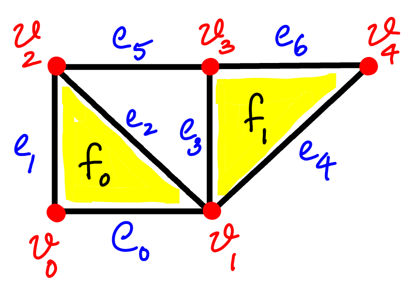
$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

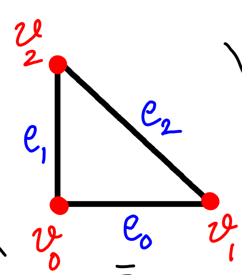
The word complex as used here is different from a simplicial complex. At the same time, the chain complex is indeed an abstract simplicial complex, with elements connected by boundary homomorphisms.

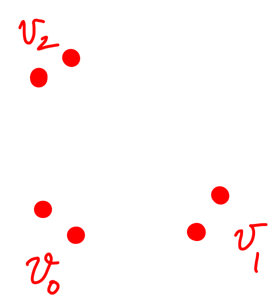
Cycles

A **p-cycle** is a p-chain with empty boundary

For instance, consider the 1-chain $\bar{z}_1$ of K:

K: 

$\partial_1 \Bigg($ 

$\Bigg) =$ 

$= \emptyset$,

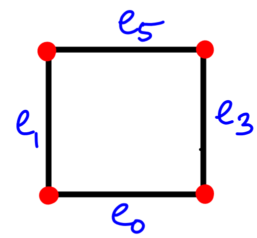
$\bar{z}_1$

with addition over $\mathbb{Z}_2$.

Hence $\bar{z}_1$ is a 1-cycle.

Alternatively, a p-chain $\bar{c}$ is a p-cycle if $\partial \bar{c} = 0$.

Here is another 1-cycle : $\bar{z}_2$:



Since ∂ commutes with $+$, the p-cycles of K form a group, denoted by Z_p or $Z_p(K)$. Z_p is a subgroup of C_p. Also, $Z_p = \ker \partial_p$, i.e., Z_p is the kernel of the p^{th} boundary homomorphism. Notice that $\partial_p \bar{z} = 0$ for $\bar{z} \in Z_p$. Also, just as C_p is, Z_p is an Abelian group.

The surface of a tetrahedron (made of four triangles) is a 2-cycle. Notice that its boundary is empty.

So, all cycles are also chains, but not the other way usually. But for $p=0$, $\partial_0 v_j = 0$, i.e., the boundary of a vertex is empty (by definition). Hence every 0-chain is also a 0-cycle, i.e., $Z_0 = C_0$.

But typically, $Z_p \subset C_p$ for $p \geq 1$.