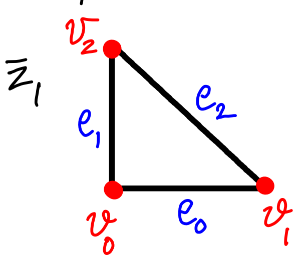
MATH 529: Lecture 16 (03/05/2026)

Today: * boundaries
* fundamental lemma of homology
* homology groups

Boundaries A **p-boundary** $\bar{b}$ is a p-chain that is the boundary of some $(p+1)$-chain $\bar{d}$, i.e., $\bar{b} = \partial_{p+1} \bar{d}$ for $\bar{d} \in C_{p+1}$.

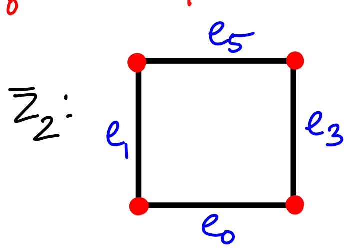
Again, since ∂ commutes with $+_2$ (or $+$) we get a group of p-boundaries B_p (or $B_p(K)$). B_p is a subgroup of Z_p, and of C_p.

B_p is the image of $\partial_{p+1} : C_{p+1} \to C_p$. $B_p = \text{im}\, \partial_{p+1}$. B_p is abelian.

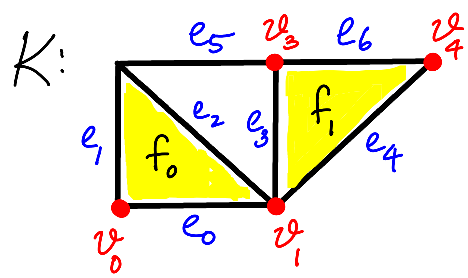


$\bar{z}_1 = \partial \bar{f}_0$, and hence is a 1-boundary.

2-chain made of triangle f_0



But $\bar{z}_2$ is not a 1-boundary.



We have $B_p \subset Z_p \subset C_p$ (in general).

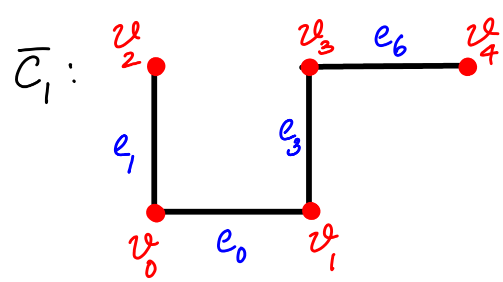
So, in summary, p-cycle $\bar{z}$: $\partial \bar{z} = 0$; $Z_p = \ker \partial_p$.

p-boundary $\bar{b}$: $\bar{b} = \partial_{p+1} \bar{d}$; $B_p = \text{im}\, \partial_{p+1}$.

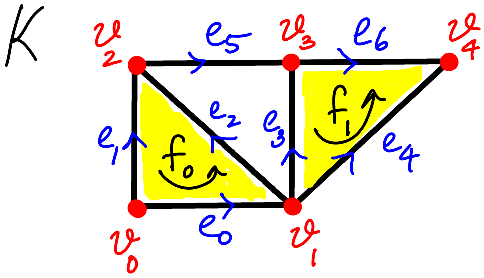
Examples

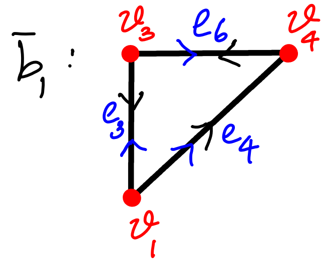
v_2 v_4 is a 0-boundary,

as it is $\partial_1 \bar{c}_1$.



Now consider oriented K (over $\mathbb{Z}$):





$$\bar{b} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

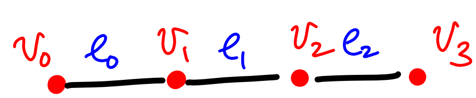
$\bar{b}_1$ is a 1-boundary, as $\bar{b}_1 = \partial_2 \bar{f}_1$ → elementary 2-chain of f_1

Let's try to enumerate how many p-cycles and p-boundaries are there for general p. We want to study cycles that are not boundaries.

$p=0$ case (over $\mathbb{Z}_2$)

$\bar{c}$, a 1-chain, is a collection of edges. $\partial_1 \bar{c}$ gives end points of edges with duplicate end points canceled in pairs, leaving an even number of distinct v_j's.

If K is connected, for every set of even number of vertices, we can find paths made of edges that connect the vertices such that the 1-chain made of these edges has as its boundary the collection of vertices.

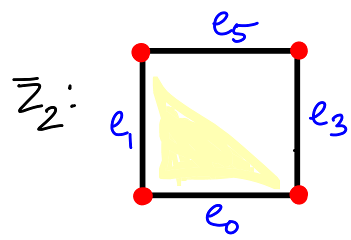


e.g., $\partial_1(e_0 + e_2) = \{v_0, v_1, v_2, v_3\}$.

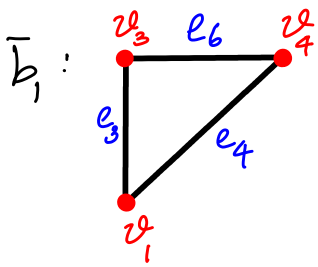
Thus, every set of even number of vertices (v_j's) is a 0-boundary, while every odd set is not. Hence, if K is connected, exactly half of the 0-cycles are 0-boundaries.

But we typically cannot make similar statements about counts of p-cycles and p-boundaries for $p \geq 1$.

We want to characterize cycles that are not boundaries, as they capture holes, e.g., as $\bar{z}_2$ here.



An observation: Consider $\bar{b}_1 = \partial_2 \bar{f}_1$.



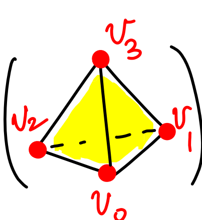
$\partial \bar{b}_1 = 0$ (as each v_j cancels in pairs).

In other words, $\partial_1 \partial_2 f_1 = 0$. This result holds in general!

Fundamental Lemma of Homology $\partial_p \partial_{p+1} \bar{d} = 0 \ \forall p \in \mathbb{Z}$.

In words, boundary of a boundary is empty.

Proof (over $\mathbb{Z}_2$) For each $(p+1)$-simplex τ, $\partial_p \partial_{p+1} \tau = 0$, as $\partial_{p+1}\tau$ consists of all p-faces of τ. Each $(p-1)$-face of τ belongs to exactly two p-faces.

$\partial_2\Bigg(\partial_3\Bigg($  $\Bigg)\Bigg) = \partial_2\big(\text{union of 4 triangles}\big) = 0,$

tetrahedron

as each edge is shared by exactly two triangles.

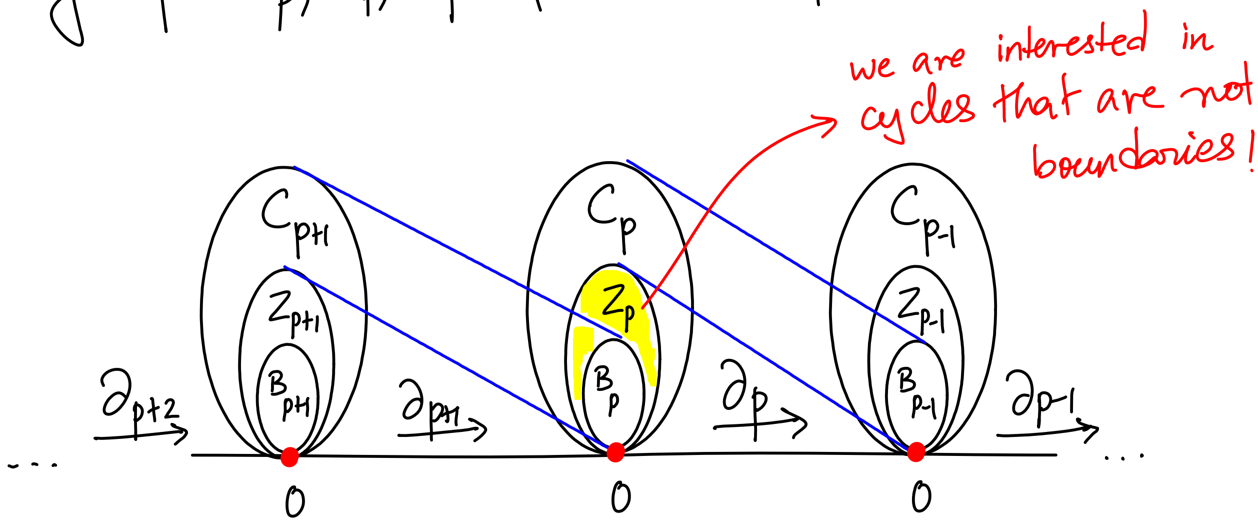
$$\partial_p\Big(\partial_{p+1}\tau = \partial_{p+1}\big([v_0 \cdots v_{p+1}]\big)\Big) = \partial_p\Big(\sum_{j=0}^{p+1} [v_0 \cdots \hat{v}_j \cdots v_{p+1}]\Big)$$

WLOG, the $(p-1)$-simplex $[v_0 \cdots v_{p-1}]$ is a face of $[v_0 \cdots v_{p-1} v_p]$ and $[v_0 \cdots v_{p-1} v_{p+1}]$.

The result holds over $\mathbb{Z}$ as well (over any ring, in fact). We must consider induced orientations when taking ∂_{p+1} and ∂_p.

A p-boundary is also a p-cycle. Hence B_p is a subgroup of Z_p.

The groups C_p, Z_p, B_p for various p are related as follows:



Homology Groups

Since B_p is a subgroup of Z_p, we can take quotients.

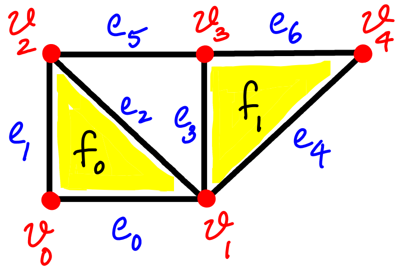
Def The **p-th homology group** is the p-th cycle group modulo the p-boundary group, $H_p = Z_p / B_p$.

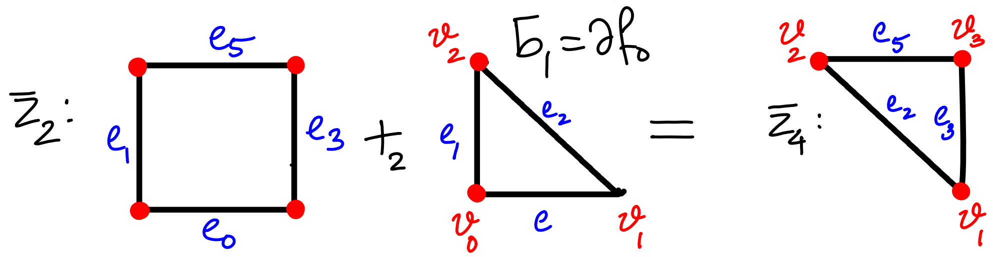
H_p has the classes of cycles that are not boundaries.

Each element of $H_p(K)$ is obtained by adding p-boundaries to given p-cycles (in their class), i.e., $z + B_p$, where $z \in Z_p$.

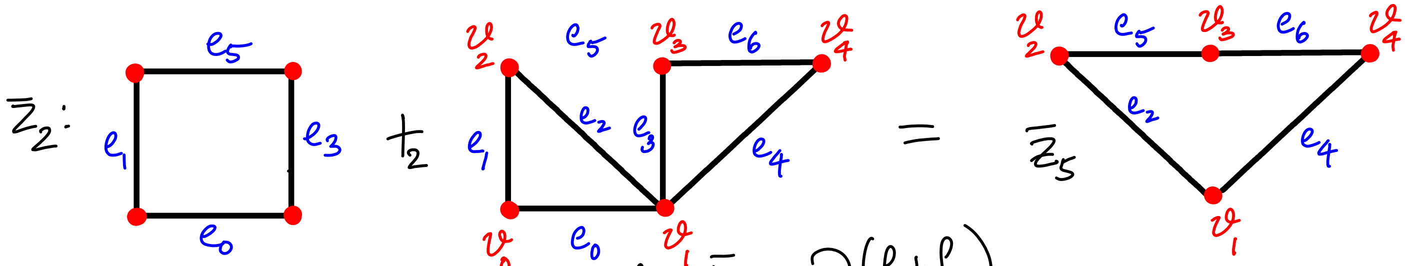
$z + B_p$ is a coset of B_p in Z_p.

Example





$\bar{z}_2$ and $\bar{z}_4$ are cycles going around the same hole.



$\bar{b}_3 = \partial_2(f_0 + f_1)$

$\bar{z}_5$ also goes around the same hole as $\bar{z}_2$ (and $\bar{z}_4$).

We could use any one cycle going around the hole ($\bar{z}_2, \bar{z}_4, \bar{z}_5$) as a representative of $H_1(K)$.

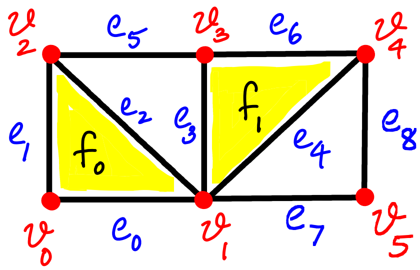
In general, another cycle $\bar{z}' = \bar{z} + \bar{b}$ for $\bar{z}$ in $H_p(K)$ and $\bar{b} \in B_p$ is in the same class as $\bar{z}$, i.e.,
$\bar{z}' + B_p = \bar{z} + B_p$ (as $\bar{b} + B_p = B_p$ itself). This is a class in $H_p(K)$, and any two cycles in this class are said to be **homologous**, written as $\bar{z} \sim \bar{z}'$.

In this setting, addition of classes is well-defined:
$(\bar{z} + B_p) + (\bar{z}' + B_p) = (\bar{z} + \bar{z}' + B_p)$, independent of the particular representatives $\bar{z}$ and $\bar{z}'$.

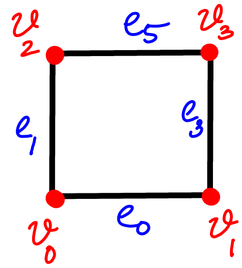
Thus H_p is indeed a group, and is abelian.

Another example

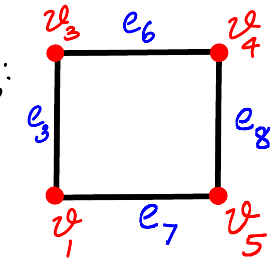
K_2:



$\bar{z}_2$:



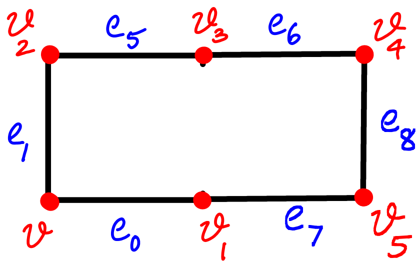
$\bar{z}_6$:



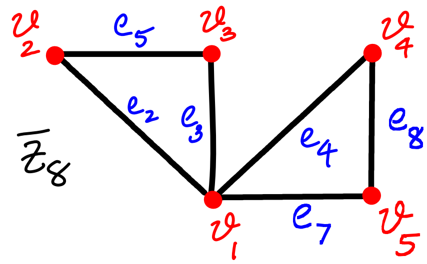
Notice $\bar{z}_6 \nsim \bar{z}_2$, as we cannot get $\bar{z}_6 = \bar{z}_2 + \bar{b}$ for any $\bar{b} \in B_1$.

Now consider two more cycles $\bar{z}_7$ and $\bar{z}_8$ as shown.

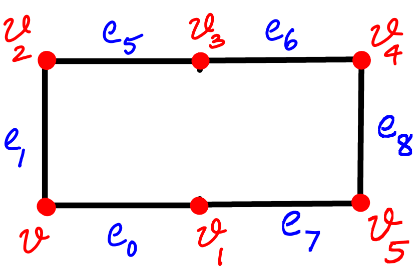
$\bar{z}_7$:



$\bar{z}_8$

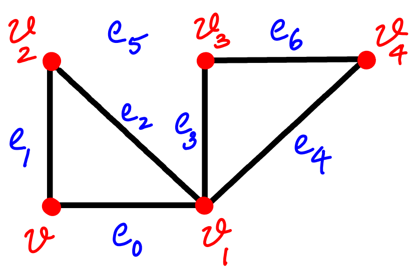


$\bar{z}_7 \sim \bar{z}_8$ as



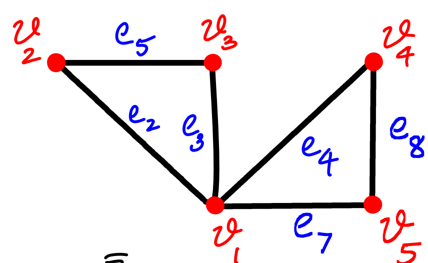
$\bar{z}_7$

$+_2$



$\bar{b}_3$

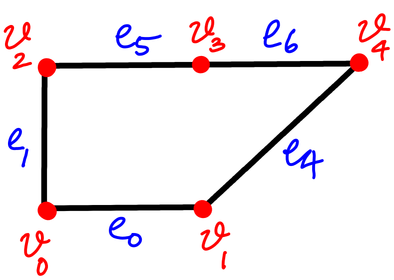
$=$



$\bar{z}_8$

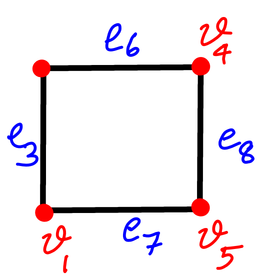
Now, consider $\bar{z}_9$ and $\bar{z}_6$. Notice that $\bar{z}_9 \nsim \bar{z}_6$. But

$\bar{z}_9$:

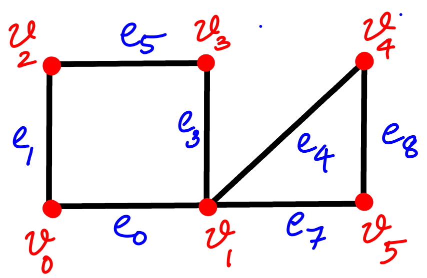


$+$

$\bar{z}_6$:



$=$



$\bar{z}_{10}$

$\bar{z}_{10} = \bar{z}_9 + \bar{z}_6$. $\quad \bar{z}_{10} \nsim \bar{z}_9$, but $\bar{z}_{10} \sim \bar{z}_8$

To describe $H_1(K_2)$ completely, we could present $[\bar{z}_2]$ and $[\bar{z}_6]$, or equivalently, $[\bar{z}_2]$ and $[\bar{z}_7]$, or $[\bar{z}_6]$ and $[\bar{z}_7]$.

homology class of $\bar{z}_2$

Intuitively, since K_2 has two holes, we expect $H_1(K_2)$ to have two classes.