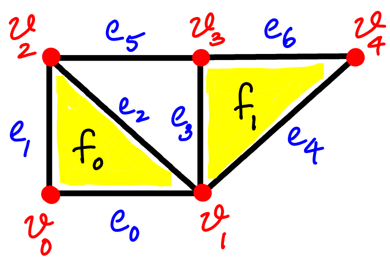
MATH 529: Lecture 18 (03/12/2026)

Today: * $[\partial_p]$ matrix
* Smith Normal Form

Recall $[\partial_p]$: $m \times n$ $\{0,1\}$-matrix (over $\mathbb{Z}_2$) when K has m $(p-1)$-simplices and n p-simplices.

$[\partial_p]_{ij} = \{1$ if $\tau_i \leq \sigma_j$, 0, otherwise$\}$

Example

K: 

$$[\partial_0] = \begin{array}{c} \\ 1 \end{array} \begin{array}{c} \begin{matrix} v_0 & v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

default, as there are no (-1)-dimensional simplices

$$[\partial_2] = \begin{array}{c|cc} & f_0 & f_1 \\ \hline e_0 & 1 & 0 \\ e_1 & 1 & 0 \\ e_2 & 1 & 0 \\ e_3 & 0 & 1 \\ e_4 & 0 & 1 \\ e_5 & 0 & 0 \\ e_6 & 0 & 1 \end{array}$$

$$[\partial_1] = \begin{array}{c|ccccccc} & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_0 & 1 & 1 & & & & & \\ v_1 & 1 & & 1 & 1 & 1 & & \\ v_2 & & 1 & 1 & & & 1 & \\ v_3 & & & & 1 & & 1 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

(entries not listed are zeros).

A collection of columns represents a p-chain, and a collection of rows represents a $(p-1)$-chain. Given a p-chain $\bar{c} = \sum_{j=1}^{n} a_j \sigma_j$, $a_j \in \{0,1\}$, its p-boundary $\partial_p \bar{c}$ is given by the matrix-vector product $[\partial_p]\bar{c}$, i.e., the sum of the corresponding columns with weights a_j gives its p-boundary.

Similarly a collection of rows represents a $(p-1)$-chain, and their sum gives the $(p-1)$-coboundary. Coboundary and cohomology is a dual concept to boundary and homology.

Since every p-chain can be written as $\bar{c} = \sum_{j=1}^{n} a_j \sigma_j$, the columns of $[\partial_p]$ generate C_p. Similarly, rows of $[\partial_p]$ generate C_{p-1}.

To compute z_p, b_p, β_p for all p, we will use operations similar to Gaussian elimination in linear algebra. Here we perform both elementary row operations (EROs) and elementary column operations (ECOs).

EROs on $A_{m \times n}$: $\quad R_i \rightleftarrows R_j$ (swap rows i and j)

over $\mathbb{R}$, $c \in \mathbb{R}$, and $\quad R_i \leftarrow R_i + cR_j$ ($R_i + cR_j$, replacement)

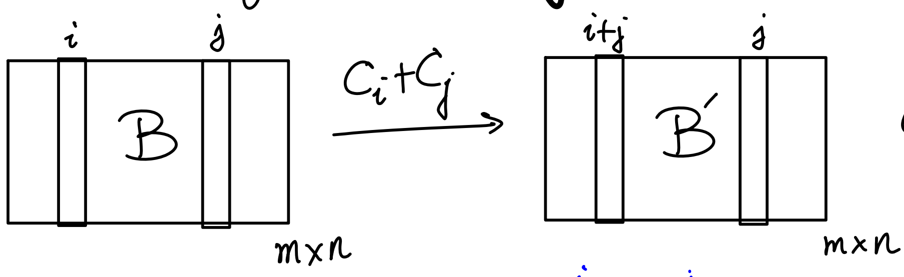
over $\mathbb{Z}$, $c \in \mathbb{Z}$. $\quad R_i \leftarrow cR_i$, $c \neq 0$ (cR_i, scaling)

Over $\mathbb{Z}_2$, $c \in \{0,1\}$, and only swap and replacement need to be considered. We write $R_i \rightleftarrows R_j$, $R_i + R_j$, in short.

ECOs are defined similarly: $C_i \rightleftarrows C_j$, $C_i + C_j$ (over $\mathbb{Z}_2$).

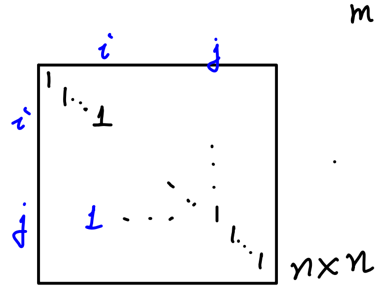
ECOs can be represented by multiplication on the right by the corresponding **elementary matrix**.

e.g.,

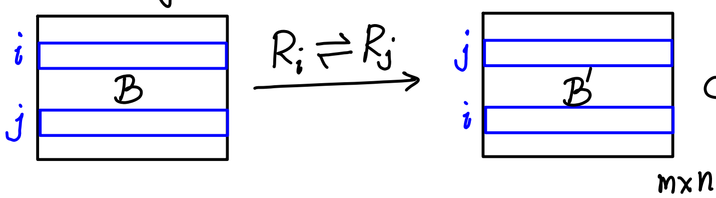


can be written as

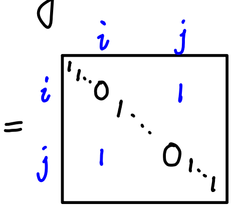
$B' = BV$, where $V =$



Similarly, EROs can be represented by multiplication the left by an elementary matrix.



can be written as $B' = UB$ for $U =$



Smith Normal Form (SNF)

This is the analogue of reduced row echelon form (RREF) of a matrix, from linear algebra. SNF is defined over $\mathbb{Z}$, in general, and accounts for both EROs and ECOs.

Def A matrix $B \in \mathbb{Z}^{m \times n}$ is in **Smith normal form** if

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \begin{matrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_\ell \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

where

Each '0' is a submatrix of all zeros.

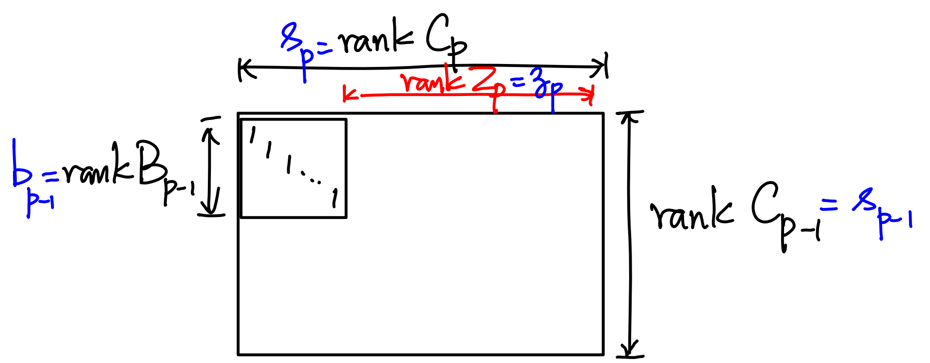
$d_i \in \mathbb{Z}$, $d_i \geq 1$ and $d_1 | d_2 | d_3 | \cdots | d_\ell$.

"divides": $a|b$ means a divides b.

When working over $\mathbb{Z}_2$, we will get

$$\mathrm{SNF}([\partial_p]) = \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right].$$

In more detail, we will have the following structure for $\mathrm{SNF}([\partial_p])$.

$\mathrm{SNF}([\partial_p]) =$ 

We get z_p and b_{p-1} from $\mathrm{SNF}([\partial_p])$. To compute β_p, we need to know also b_p, which is obtained from $\mathrm{SNF}([\partial_{p+1}])$.

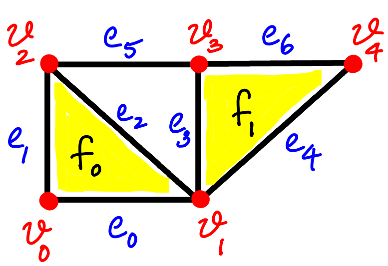
Overall, we can write $SNF([\partial_p]) = U_{p-1}[\partial_p]V_p$, where U_{p-1} captures all EROs performed, and V_p captures all ECOs performed. We also get information about bases of Z_p and B_{p-1} from U_{p-1} and V_p.

More specifically, a basis for the cycle group Z_p is encoded in the last z_p columns of V_p. Similarly, a basis for the boundary group B_{p-1} is encoded in U_{p-1} — in the first b_{p-1} columns of U_{p-1}^{-1}, to be exact.

Thus we can reduce each $[\partial_p]$ to SNF, and in that process, compute all β_p, and also identify bases for Z_p and B_p. In the following illustration, we keep track of the bases as we perform the corresponding EROs or ECOs on $[\partial_p]$. Later on, we describe an algorithm that does all the operations in a unified manner.

Example

We consider our favorite example and reduce each $[\partial_p]$ to SNF for $p = 0, 1, 2$.

K: 

$$[\partial_0] = \begin{array}{c} \\ 1 \end{array} \begin{array}{ccccc} v_0 & v_1 & v_2 & v_3 & v_4 \\ \hline \end{array}$$

$$[\partial_0] = \begin{array}{c|ccccc} & v_0 & v_1 & v_2 & v_3 & v_4 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

default, as there are no (−1)-dimensional simplices

$$[\partial_1] = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & & & & & \\ 1 & & 1 & 1 & 1 & & \\ & 1 & 1 & & & 1 & \\ & & & 1 & & 1 & 1 \\ & & & & 1 & & 1 \end{bmatrix} \end{array}$$

(entries not listed are zeros).

$$[\partial_2] = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cc} f_0 & f_1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

We reduce each $[\partial_p]$ to SNF.

We consider $[\partial_0]$ and $[\partial_2]$ first, since they are simpler compared to $[\partial_1]$ (we consider that the last).

$$[\partial_0] = \begin{array}{cc} & \begin{array}{ccccc} v_0 & v_1 & v_2 & v_3 & v_4 \end{array} \\ 1 & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{for } j=2,\dots,5]{C_j + C_1} \begin{array}{cc} & \begin{array}{ccccc} v_0 & v_1+v_0 & v_2+v_0 & v_3+v_0 & v_4+v_0 \end{array} \\ 1 & \begin{bmatrix} 1 & \underbrace{0 \quad 0 \quad 0 \quad 0}_{z_0=4} \end{bmatrix} \end{array}.$$

But $s_0 = 5 = z_0$!

We had noticed previously that every 0-chain is also a 0-cycle, as vertices have empty boundaries. So, we should have got $s_0 = z_0 = 5$ here! We will address this discrepancy in $p=0$ soon. For now, work with $z_0 = s_0 = 5$.

A basis for $B_0(K) = \{v_j + v_0 \mid j = 1,2,3,4\}$. A collection of even pairs of vertices, such that any even set of vertices can be written as a combination of these pairs.

Recall that every 0-chain is also a 0-cycle since it has no boundary (when we work over $\mathbb{Z}_2$). And every 0-chain with an even number of vertices is a 0-boundary (assuming K is connected).

Adding any one vertex (by itself) to the above basis will give another basis for Z_0 (and hence C_0), e.g., $\{v_j + v_0|_{j=1}^{4}, v_0\}$. Recall that $\{v_j\}_{j=0}^{4}$ gives the default (elementary chain) basis for Z_0 (and for C_0).

Notation: $e_{ijk} \equiv e_i + e_j + e_k$

$$[\partial_2] = \begin{array}{c|cc} & f_0 & f_1 \\ \hline e_0 & 1 & 0 \\ e_1 & 1 & 0 \\ e_2 & 1 & 0 \\ e_3 & 0 & 1 \\ e_4 & 0 & 1 \\ e_5 & 0 & 0 \\ e_6 & 0 & 1 \end{array} \xrightarrow[\substack{R_5+R_4 \\ R_7+R_4}]{\substack{R_2+R_1 \\ R_3+R_1}} \begin{array}{c|cc} & f_0 & f_1 \\ \hline e_0 & 1 & 0 \\ e_{01} & 0 & 0 \\ e_{02} & 0 & 0 \\ e_3 & 0 & 1 \\ e_{34} & 0 & 0 \\ e_5 & 0 & 0 \\ e_{36} & 0 & 0 \end{array} \xrightarrow{R_2 \rightleftarrows R_4} \begin{array}{c|cc} & f_0 & f_1 \\ \hline e_0 & 1 & 0 \\ e_3 & 0 & 1 \\ e_{02} & 0 & 0 \\ e_{01} & 0 & 0 \\ e_{34} & 0 & 0 \\ e_5 & 0 & 0 \\ e_{36} & 0 & 0 \end{array} \quad \cancel{0}$$

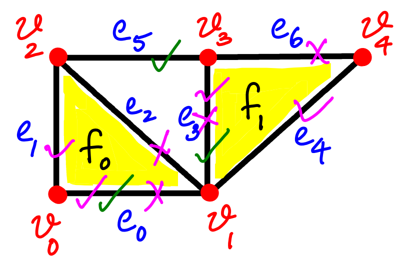
there are no 2-cycles here!

no 0-matrix to the right!

$z_2 = 0, \; b_1 = 2$

Basis for $C_2(K) = \{f_0, f_1\}$.

K:



A basis for $C_1(K) = \{e_0, e_3, e_5, e_0+e_1, e_0+e_2, e_3+e_4, e_3+e_6\}$.

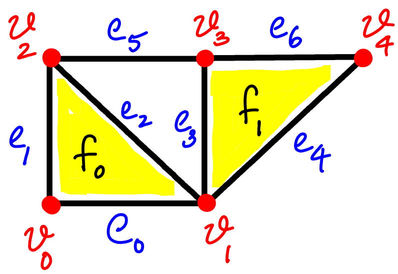
Notice that we have found also a basis for $B_1(K)$ – consisting of the boundaries of f_0 and f_1.

We are numbering the rows and columns starting from 1. As we proceed further into the reduction (to SNF), the labels for the rows/columns could become more complicated.

Similar notation for column labels: $c_{ijk} \equiv c_i + c_j + c_k$.

Let's look at $[\partial_1]$ now.

K:



$$[\partial_1] = \begin{array}{c|ccccccc|} & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_0 & 1 & 1 & & & & & \\ v_1 & 1 & & 1 & 1 & 1 & & \\ v_2 & & 1 & 1 & & & 1 & \\ v_3 & & & & 1 & & 1 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

$$\xrightarrow[\text{and then } C_2+C_1]{R_2+R_1} \begin{array}{c|ccccccc|} & e_0 & e_{01} & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_0 & 1 & 0 & & & & & \\ v_{01} & 0 & 1 & 1 & 1 & 1 & & \\ v_2 & & 1 & 1 & & & 1 & \\ v_3 & & & & 1 & & 1 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

$$\xrightarrow[\substack{\text{then } C_j+C_2 \\ j=3,4,5}]{R_3+R_2} \begin{array}{c|ccccccc|} & e_0 & e_{01} & e_{012} & e_{013} & e_{014} & e_5 & e_6 \\ \hline v_0 & 1 & & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & & \\ v_{012} & & 0 & 0 & 1 & 1 & 1 & \\ v_3 & & & & 1 & & 1 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

$$\xrightarrow{C_3 \rightleftarrows C_4} \begin{array}{c|ccccccc|} & e_0 & e_{01} & e_{013} & e_{012} & e_{014} & e_5 & e_6 \\ \hline v_0 & 1 & & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & & \\ v_{012} & & 0 & 1 & 0 & 1 & 1 & \\ v_3 & & & 1 & & & 1 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

$$\xrightarrow[\substack{\text{then} \\ C_5+C_3 \\ C_6+C_3}]{R_4+R_3} \begin{array}{c|ccccccc|} & e_0 & e_{01} & e_{013} & e_{012} & e_{34} & e_{0135} & e_6 \\ \hline v_0 & 1 & & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & & \\ v_{012} & & 0 & 1 & 0 & 0 & 0 & \\ v_{0123} & & & 0 & & 1 & 0 & 1 \\ v_4 & & & & & 1 & & 1 \end{array}$$

$$\xrightarrow[\substack{\text{then} \\ R_5+R_4 \\ \text{then} \\ C_7+C_4}]{C_5 \rightleftarrows C_4} \begin{array}{c|ccccccc|} & e_0 & e_{01} & e_{013} & e_{34} & e_{012} & e_{0135} & e_{346} \\ \hline v_0 & 1 & & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & & \\ v_{012} & & 0 & 1 & 0 & 0 & 0 & \\ v_{0123} & & & 0 & 1 & & 0 & 0 \\ v_{01234} & & & & 0 & & & 0 \end{array} = \operatorname{SNF}([\partial_1]).$$

$\rightarrow \{\underbrace{e_0+e_1+e_2,\ e_3+e_4+e_6,}_{\text{generates } Z_1(K)}\ \underbrace{e_0+e_1+e_3+e_5}_{\text{generates } H_1(K)}\}$

basis for $B_1(K)$.

$z_1 = 3, \quad b_0 = 4$

Thus we get $\beta_1 = z_1 - b_1 = 3-2 = 1$, and $\beta_0 = z_0 - b_0 = 5-4 = 1$, and both of these numbers agree with intuition (1 hole and 1 component).

$\beta_p = 0 \quad \forall p \geqslant 2$.