

MATH 529: Lecture 19 (03/24/2026)

Today: * Reduction algorithm for $[a_p]$
* reduced homology
* relative homology

A final observation on the matrix reduction example...

Diagram of a square with a hole, labeled K . Vertices are $v_0, v_1, v_2, v_3, v_4, v_5$. Edges are $e_0, e_1, e_2, e_3, e_4, e_5, e_6$. Faces are f_0 and f_1 . A hole is present in the center.

SNF $([a_1]) =$

	e_0	e_{01}	e_{013}	e_{24}	e_{012}	e_{0135}	e_{346}
v_0	1				0		
v_{01}		1	0	0	0		
v_{012}			1	0	0	0	
v_{0123}				1		0	0
v_{01234}					0		0

The labels for the Z_1 columns in SNF $([a_1])$: e_{012} e_{0135} e_{346}

The column labels of the last three (zero) columns of SNF $([a_1])$ gives a basis for Z_1 . They are e_{012} , e_{0135} , and e_{346} , i.e., $\{\{e_0, e_1, e_2\}, \{e_0, e_1, e_3, e_5\}, \{e_3, e_4, e_6\}\}$. These three loops consist of ∂f_0 , ∂f_1 , and $\{e_0, e_1, e_3, e_5\}$, which captures the hole. $\{e_0, e_1, e_3, e_5\}$ can be used to represent H_1 here.

Notice that we may not obtain the actual "boundary of the hole" — we're guaranteed to get one loop around the hole, but it may not be the shortest or tightest loop!

Algorithm for SNF over \mathbb{Z}_2 of $B \in \{0,1\}^{m \times n}$

Here is the main block:

```

void REDUCE(i)
  if  $\exists j \geq i, k \geq i$  with  $B_{jk} = 1$  then
     $R_j \Leftrightarrow R_i; C_k \Leftrightarrow C_i$ 
    for  $h = i+1$  to  $m$ 
      if  $B_{hi} = 1$  then
         $R_h + R_i$ 
      endif
    endfor
    for  $l = i+1$  to  $n$ 
      if  $B_{il} = 1$  then
         $C_l + C_i$ 
      endif
    endfor
    REDUCE(i+1);
  endif

```

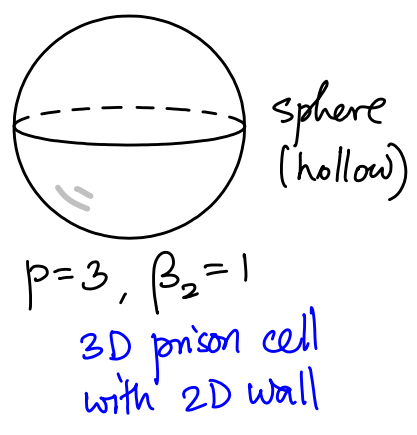
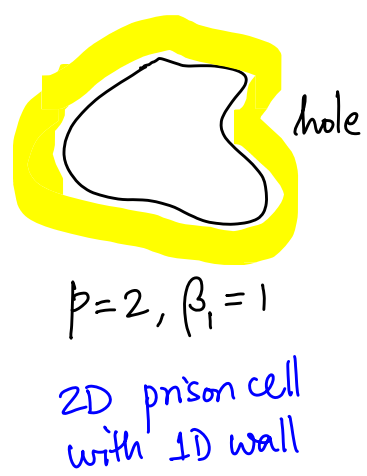
} $B_{ii} = 1$ after this step (this step allows the possibility of $B_{ii} = 1$ to start with)
 } Zero out 1's below B_{ii} using replacement ERDs
 } zero out 1's to the right of B_{ii} using replacement ECDs
 → proceed to next smaller block to right and below

We could initiate (as identity matrices) the elementary matrices $U_{m \times m}$ and $V_{n \times n}$, as well as U^{-1} , and update them along the way to identify bases for $\mathbb{Z}_p, \mathbb{B}_{p-1}$, etc.

You will get the chance to implement this reduction algorithm in Hw5...

Reduced Homology groups

To take care of the discrepancy at $p=0$ ($\mathbb{Z}_0 = \mathbb{Z}_0$ or not?)
IDEA: "A prison cell in p -dimensions has $(p-1)$ -dimensional walls."



We count $\beta_1=1$ for the hole when the 2D solid patch is missing in the middle. Similarly, when the solid 3D object/space is missing from the middle of a sphere, we set $\beta_2=1$.

Note that we're removing a p -ball in each case to make the prison cell (2-ball \approx disc, 3-ball \approx solid 3D patch, 1-ball \approx edge).

.....
 $\beta_0=1?$

Under this idea, we would count $\beta_0=1$ when we have two vertices (with the "edge in between missing", capturing a "hole"). But if β_0 is to count the number of connected components, $\beta_0=2$ here.

To resolve this confusion, we add the augmentation map $\epsilon: C_0 \rightarrow \mathbb{Z}_2$, where $\epsilon(v_j)=1$ for each $v_j \in K$.

$$\dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}_2 \xrightarrow{0} 0$$

We push off the "boundary effect" one step more so that the general structure holds for $p=0$ as well!

\leftarrow new, as compared to the original chain complex

Notice that $\epsilon \circ \partial_1 = 0$, similar to $\partial_p \partial_{p+1} = 0 \neq p$, as each edge has two vertices, and both vertices have value 1 under ϵ and $1 \pm 1 = 0$.

Under ϵ , a 0-cycle must have an even number of vertices. As the 1-boundary of an edge (1-simplex) will have two vertices, and each of them has a value of 1 under ϵ , giving $1 \pm 1 = 0$.

The new chain complex with homomorphisms ∂_p along with ϵ gives rise to reduced homology groups $\tilde{H}_p(K)$.

We also get the reduced Betti numbers, $\tilde{\beta}_p$, as the corresponding ranks: $\tilde{\beta}_p = \text{rank } \tilde{H}_p$.

Assuming $K \neq \emptyset$, $\tilde{\beta}_p = \beta_p$ for $p \geq 1$, and $\tilde{\beta}_0 = \beta_0 - 1$.

A similar augmentation map is defined for homology over \mathbb{Z} . $\epsilon(v_j) = 1$ for vertex v_j .

$$\dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0} 0 \rightarrow \dots$$

$\epsilon \partial_1 = 0$ here as well.

$$\epsilon \left(\partial_1 \left[\begin{array}{c} e \\ \bullet \quad \bullet \\ v_0 \quad v_1 \end{array} \right] \right) = \epsilon(v_1 - v_0) = 1 - 1 = 0.$$

Relative Homology

One way to extend the concepts of homology from finite simplicial complexes to pairs of spaces.

Let K be a simplicial complex and $K_0 \subseteq K$ be a subcomplex.

We can talk about relative chain groups as

$$C_p(K, K_0) = C_p(K) / C_p(K_0),$$

i.e., as the quotient of the chain groups in K and K_0 .

IDEA: Divide $C_p(K)$ into cosets of p -chains that differ in p -simplices in K_0 , but not over $K - K_0$.

We can consider ∂_p from $C_p(K, K_0)$ to $C_{p-1}(K, K_0)$ as the homomorphism induced by ∂_p originally defined on K .

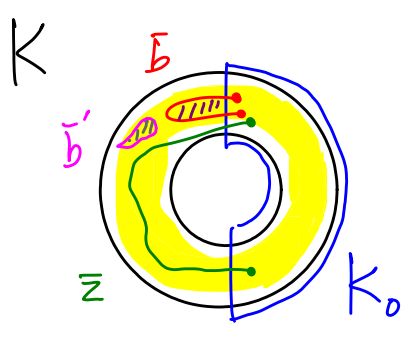
$$\partial_p : C_p(K, K_0) \rightarrow C_{p-1}(K, K_0).$$

These boundary maps also satisfy $\partial_p \circ \partial_{p+1} = 0$. Hence we can define cycles, boundaries, and homology groups in the relative setting.

$$Z_p(K, K_0) = \ker \partial_p, \quad B_p(K, K_0) = \text{im } \partial_{p+1}, \quad \text{and}$$

$$H_p(K, K_0) = Z_p(K, K_0) / B_p(K, K_0).$$

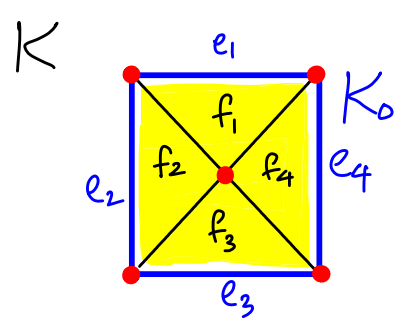
Examples Let K be an annulus, and K_0 be half of it.



\bar{z} is a 1-cycle in K/K_0 , i.e., $\bar{z} \in Z_1(K, K_0)$.
 So is \bar{b} . But \bar{b} is also a 1-boundary in K/K_0 , i.e., $\bar{b} \in B_1(K, K_0)$.
 Similarly, $\bar{b}' \in B_1(K, K_0)$, but is also in $B_1(K)$.

Intuitively, we consider everything in K_0 as "trivial", or as "reduced to empty". For instance, $\partial \bar{z}$ above is the set of its two end points, which are both in K_0 . Hence, $\partial \bar{z}$ is considered empty in K/K_0 , and hence \bar{z} is a relative cycle.

Here is another example over \mathbb{Z} . Let K_0 be the four boundary edges of K , which is made of the four triangles making a square disc.



$K_0 = \{e_1, e_2, e_3, e_4, \text{ and four outer vertices}\}$.

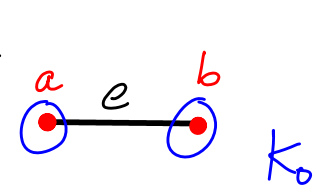
$\bar{c} = \sum_1^4 a_j f_j$ is a general 2-chain of K

When $a_j = a \in \mathbb{Z} \forall j$, $\partial \bar{c} \in K_0$ (when $a=1$, we get all of K_0 as $\partial \bar{c}$). In this case, $\partial \bar{c}$ is not empty in K .
 Hence, \bar{c} is not a 2-cycle in K .

But $\bar{c} \in Z_2(K, K_0)$ as $\partial \bar{c}$ is contained in K_0 .

Intuitively, we "shrink" all of the boundary of \bar{c} to empty, thus "creating" a 2-sphere out of it!

Notice that K_0 need not be connected. Consider the example where K consists of the single edge $e = \bar{ab}$ (and the vertices a and b). Let $K_0 = \{a, b\}$, the two vertices. Here, e is a relative 1-cycle in (K, K_0) .



We can compute ranks of, and bases for, relative homology groups using the same concept of SNF of $[\partial_p]$.

We start off by defining the relative boundary matrix for (K, K_0) as the submatrix of $[\partial_p]$ obtained by removing the rows and columns corresponding to simplices in K_0 . The rest of the procedure is identical to the one used in the absolute, i.e., default, case.

We will use the idea of relative homology in defining some of the concepts in the upcoming lectures.