

MATH 529: Lecture 20 (03/26/2026)

Today: * persistence
* incremental algorithm for betti numbers

Persistent Homology

We will consider a filtered complex $K \phi = K^0 \subseteq K^1 \subseteq \dots \subseteq K^m = K$,

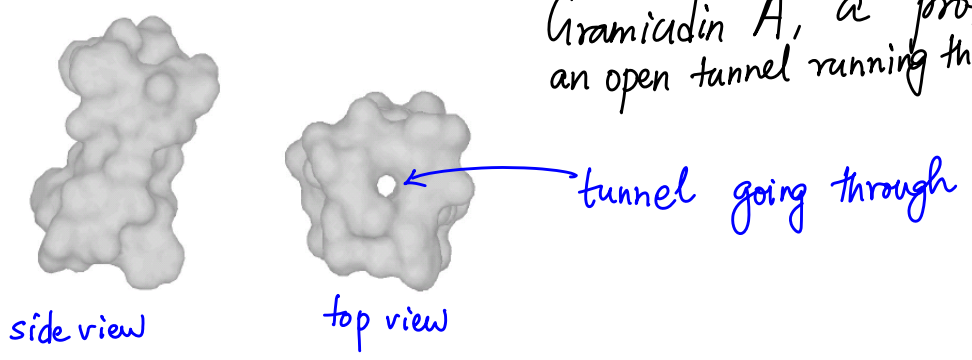
$|K| = X$, some space.

Let $\beta_k^l = \text{rank}(H_k^l)$ where $H_k^l = Z_k^l / B_k^l$ with $Z_k^l = Z_k(K^l)$, etc.

We could track β_k^l for fixed k as l varies, getting more information than obtained from any single l .

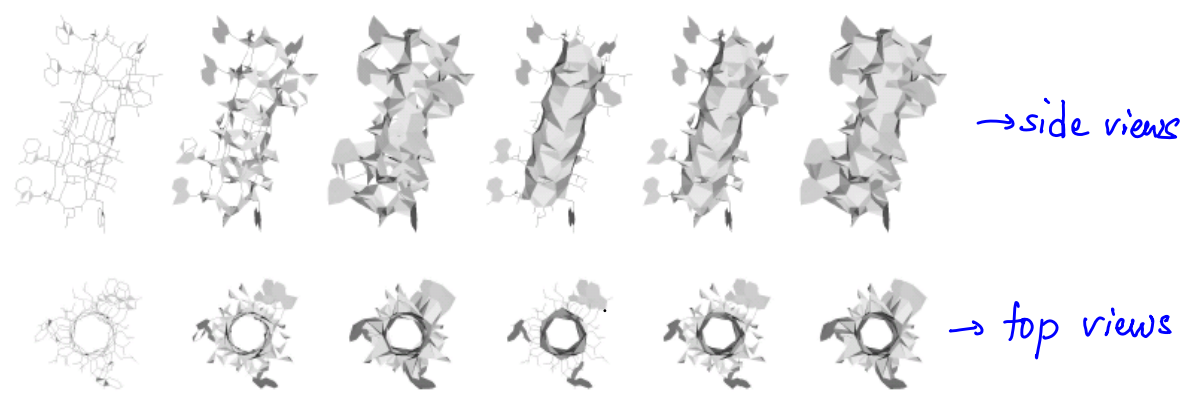
Example (from Edelsbrunner, Letscher, Zomorodian, 2002)

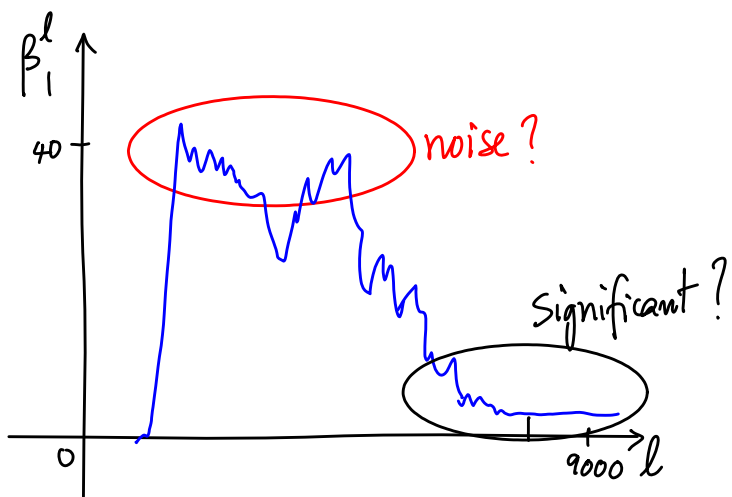
Gramicidin A, a protein with an open tunnel running through it.



Shown below are certain subcomplexes from the α -complex filtration of gramicidin.

→ with atoms as points, having different starting radii.





There appears to be lots of holes showing up early on, which all get "closed" later on. We need to separate topological noise from features.

A "significant" feature should have a long "lifetime" in the filtration. We look for non-bounding cycles that do **not** turn into boundaries "in the near future," i.e., for the next p complexes in the filtration. Thus, we look for cycles in K^l that stay non-bounding till K^{l+p} .

Def The p -persistent k^{th} homology group of K^l is

$$H_k^{l,p} = Z_k^l / (B_k^{l+p} \cap Z_k^l).$$

And $\beta_k^{l,p} = \text{rank}(H_k^{l,p})$ is the p -persistent k^{th} betti number of K^l .

Let the non-bounding k -cycle \bar{z} be created at time i (for index $l, l+1, \dots$) with the arrival of simplex σ^i . So $[\bar{z}]$, the homology class represented by \bar{z} is in H_k^i . Let σ^j arrive at time j , and turn $\bar{z}' \in [\bar{z}]$ into a boundary. The " k -dimensional hole" captured by $[\bar{z}]$ is "closed" by σ^j .

So $\bar{z}' \in B_k^j$, and $[\bar{z}]$ is now merged with some older class of cycles, i.e., it no longer exists independently. But it did exist independently for all times $i \leq t < j$, i.e., for $j-i-1$ steps.

The **persistence** of $[\bar{z}]$ (or of \bar{z} itself) is $j-i-1$, and $[i, j)$ is its **life-time** in the filtration.

σ^i is the **creator** of $[\bar{z}]$, and σ^j is the **destroyer**.

A simplex that is a creator is a positive simplex, and one that destroys is a negative simplex.

If a class has no destroyer, its persistence is $+\infty$.

Time-based persistence

$$H_k^{\lambda, \mu} = Z_k^\lambda / (B_k^{\lambda+\mu} \cap Z_k^\lambda) \quad \text{for times } \lambda, \mu \geq 0.$$

We could have the filtration growing with time (instead of adding one simplex in each step). By default, though, we will consider index-based persistence. The two notions are equivalent in some sense (under appropriate assumptions).

Incremental Algorithm for Betti numbers (Delfinado & Edelsbrunner, 1995)

The algorithm assumes K is in \mathbb{R}^3 and is torsion-free. Equivalently, K is a subcomplex of (a triangulation) of $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$.

We add one simplex at each step l to grow K . Equivalently, in the filtration of K , $K^j = K^{j-1} \cup \{\sigma^j\} \quad \forall j$.

We want to classify k -simplices for $k=0,1,2,3$ as creators or destroyers. Later on we will talk about pairing a destroyer with a creator, and use these pairings to compute persistence.

To simplify notation for this discussion, we write $L = K \cup \{\sigma\}$, instead of $K^j = K^{j-1} \cup \{\sigma^j\}$. We consider four cases, corresponding to σ being a vertex, edge, triangle, or a tetrahedron.

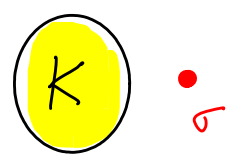
In fact, this classification is valid in general when we add a simplex σ to a simplicial complex K (i.e., not necessarily in a filtration framework).

Recall that $K^{j-1} \subseteq K^j$ in the filtration. Hence when we add a simplex σ in step j , we are guaranteed that all proper faces of σ are already present in the previous complex.

For instance, when we add $\sigma = \triangle abc$, the 0-simplices a, b, c , and the 1-simplices ab, ac, bc are already present.

A simple way to create such a filtration is to order all simplices by their dimensions, with lower dimensional simplices appearing before higher dimensional ones (we can break ties arbitrarily). Standard filtrations such as alpha or Vietoris-Rips complexes also provide such orderings once we use dimension to break ties among simplices (so, if a triangle and its faces are present at a given radius, the edges come before the face).

Case 1. σ is a 0-simplex



We get $\beta_0(L) = \beta_0(K) + 1.$

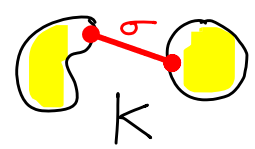
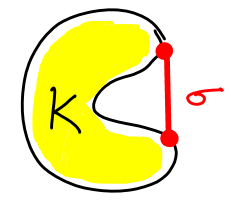
σ here adds a new class of 0-cycles. Other homology classes are not changed. Thus, 0-simplices are always creators, i.e., they are always positive.

Case 2 σ is a 1-simplex

If σ belongs to a 1-cycle in L ,

$\beta_1(L) = \beta_1(K) + 1,$

else $\beta_0(L) = \beta_0(K) - 1.$



Other β_k 's are not affected.

So, 1-simplices can be creators or destroyers.

Case 3 σ is a 2-simplex

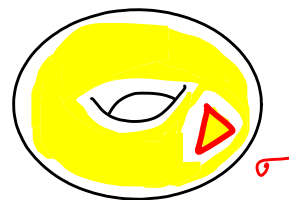
If σ belongs to a 2-cycle in L

$$\beta_2(L) = \beta_2(K) + 1;$$

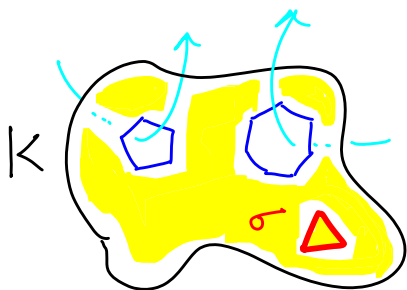
else

$$\beta_1(L) = \beta_1(K) - 1.$$

(other β_k 's remain unchanged)



all other triangles in the surface of the torus are already present, and σ just closes the final hole, thus trapping the 3D space inside.



σ closes one of the holes.

Thus, 2-simplices can be creators or destroyers, just like 1-simplices.

Case 4 σ is a 3-simplex.

$$\beta_2(L) = \beta_2(K) - 1.$$

The tetrahedron coming in closes the void formed by its four faces.

So, 3-simplices are always destroyers (or, are negative) in our setting.

Since we assumed that K is a subcomplex of a triangulation of S^3 in R^3 , we do not get any 3-cycles. So, $\beta_3(K) = 0$.

IDEA: To understand the above statement about $\beta_3=0$, we will go down in dimension by 1. Think about K being a subcomplex of S^2 and consider triangles. The only case when a triangle is positive is when it comes in to complete the surface of S^2 . But, if we further require K to be embedded in \mathbb{R}^2 , then we cannot have a 2-cycle and all triangles are negative ($\beta_2(K)=0$). The case of tetrahedra in K that is a subcomplex of a triangulation of S^3 , sitting in \mathbb{R}^3 , is similar (we get $\beta_3(K)=0$).

The idea of classifying simplices as positive and negative can be extended to arbitrary dimensions, though.

Algorithm for $(\beta_0, \beta_1, \beta_2)$ when $K \subset$ triangulation of S^3

integer³ BETTI

$\beta_0 = \beta_1 = \beta_2 = 0;$

for $j=1$ to m do filtration is $K^0 \subseteq \dots \subseteq K^m = K$
 \emptyset (m simplices in K)

$k = \dim \sigma^j;$

if σ^j belongs to a k -cycle in K^j then

$\beta_k^{++}; (\beta_k \leftarrow \beta_k + 1)$

\rightarrow non-trivial in general!

else $\beta_{k-1}^{--}; (\beta_{k-1} \leftarrow \beta_{k-1} - 1)$

endif

endfor

return $(\beta_0, \beta_1, \beta_2);$

We can understand the logic behind this algorithm using the formulae for Betti numbers. Recall,

$$\beta_k = z_k - b_k \quad (1)$$

$$\beta_{k-1} = z_{k-1} - b_{k-1} \quad (2)$$

$$s_k = z_k + b_{k-1} \quad (3)$$

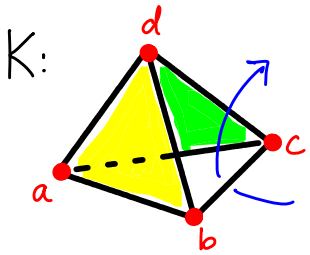
When we add a k -simplex, s_k goes up by 1, which we indicate by $s_k \uparrow$.

If $s_k \uparrow$, then (3) \Rightarrow either $z_k \uparrow$ or $b_{k-1} \uparrow$.

If $z_k \uparrow$, then (1) $\Rightarrow \beta_k \uparrow$; and

if $b_{k-1} \uparrow$, then (2) $\Rightarrow \beta_{k-1} \downarrow$.

Example



filtration: $a, b, c, d, ab, ac, ad, bc, bd, cd, abd, acd$ ($m=12$)

There is one connected component, and one hole.

So we expect $\beta_0=1, \beta_1=1$.

\hookrightarrow There are two triangles missing from the tetrahedron, but there is only one hole, as indicated by the arrow \uparrow . Also, adding one more triangle gives a disc!

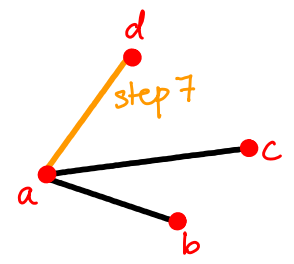
Steps of the algorithm ($\beta_0 = \beta_1 = 0, \beta_2 = 0$ to start) \rightarrow we have no positive triangles. So β_2 remains at zero.

1-4: a, b, c, d : β_0 goes $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ (all are positive simplices)

5: ab : $\beta_0 \downarrow$; ($\beta_0=3$) (ab is a negative simplex)

6: ac : $\beta_0 \downarrow$; ($\beta_0=2$) (ac is a negative simplex)

7: ad : $\beta_0 \downarrow$; ($\beta_0=1$) (ad is a negative simplex)



Will finish the example in the next lecture...