

# MATH 529: Lecture 26 (04/16/2026)

Today: \* optimality in homology  
\* homology over  $\mathbb{Z}$

## Optimality in Homology

We consider homology groups over  $\mathbb{Z}$  to define optimization problems. While  $\mathbb{Z}_2$  might be the simpler ring to consider, it turns out  $\mathbb{Z}$  opens up other possible approaches not possible with  $\mathbb{Z}_2$  in this context.

## Homology over $\mathbb{Z}$

$\mathbb{Z}$  has both  $+z$  and  $-z$  for any integer  $z$ .  $\mathbb{Z}_2$  has only 1 (no  $\pm$  signs)

We can use orientation of simplices/chains to capture the  $\pm$  signs.

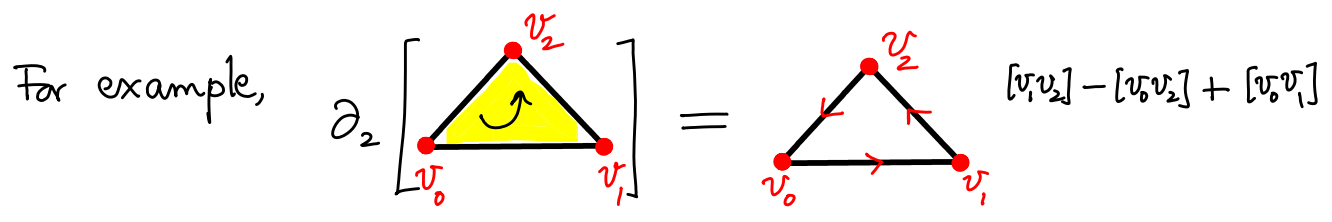
### Recall induced orientation

Let  $\sigma = [v_0, v_1, \dots, v_p]$  be a  $p$ -simplex and let  $\tau = \text{conv}\{v_0, \dots, \hat{v}_i, \dots, v_p\}$  be a  $(p-1)$ -face. The orientation induced on  $\tau$  by  $\sigma$  is the same as  $[v_0, \dots, \hat{v}_i, \dots, v_p]$  if  $i$  is even.

$\hat{v}_i$  is omitted

The boundary of a simplex is defined as follows.

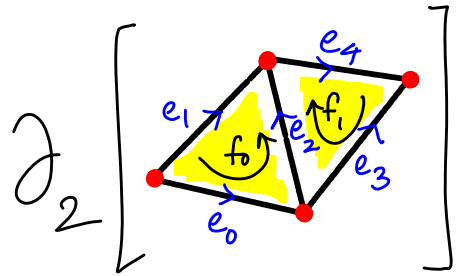
$$\partial_p \sigma = \sum_{i=0}^p (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p].$$



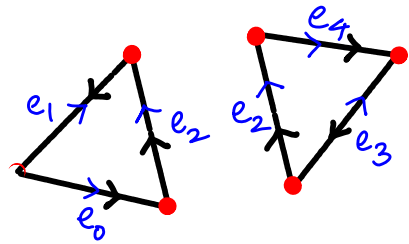
for a  $p$ -chain  $\bar{c} = \sum_{j=1}^n a_j \sigma_j$ , where  $a_j \in \mathbb{Z}$  and  $\sigma_j$ 's are  $p$ -simplices,

$$\partial_p \bar{c} = \partial \left( \sum_{j=1}^n a_j \sigma_j \right) = \sum_{j=1}^n a_j (\partial \sigma_j).$$

Choices of orientations for  $p$ -simplices become critical in boundary computations when working over  $\mathbb{Z}$ . Consider the 2-chain consisting of two triangles shown below.



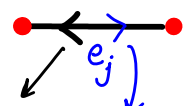
=



$$\bar{c}_1 = f_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\partial_2 \bar{c}_1 = \bar{b}_1 = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

notation:



induced orientation from face

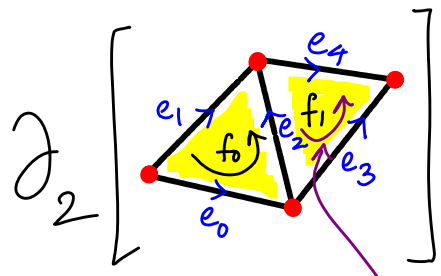
orientation of  $e_j$

Notice that  $f_0$  and  $f_1$  are not consistently oriented. Induced orientations on  $e_2$  are same, and hence  $e_2$  has weight 2 in  $\partial_2 \bar{c}_1$ .

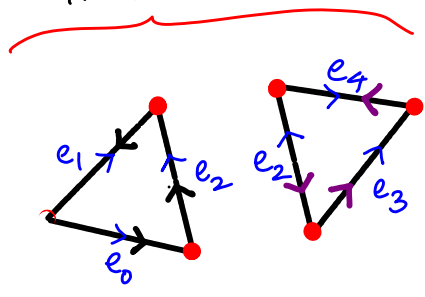
When working over  $\mathbb{Z}_2$ ,  $e_2$  will be counted twice, and hence will cancel when taking  $\partial_2 \bar{c}_1$ .

But, consider  $\partial \bar{c}_2$  where  $\bar{c}_2 = f_0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

$$\text{We get } \partial \bar{c}_2 = \bar{b}_2 = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$



=



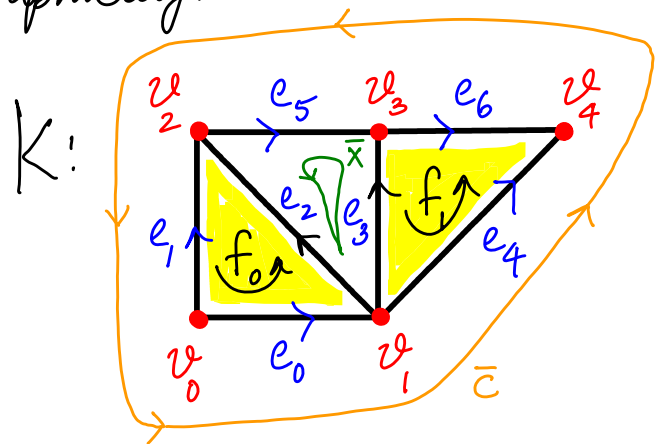
$$\bar{c}_2 = f_0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$f_0$  and  $f_1$  are consistently oriented in  $\bar{c}_2$ , and hence  $e_2$  cancels when taking boundary.  $e_2$  will cancel as long as the weights for  $f_0$  and  $f_1$  are same in absolute value, and  $f_0$  and  $f_1$  are consistently oriented.

The definitions of  $C_p, Z_p, B_p$ , and  $H_p$  are similar to what we have seen before (when we were working over  $\mathbb{Z}_2$ ). So,  $\partial_p: C_p \rightarrow C_{p-1}$  are the boundary homomorphisms with addition over  $\mathbb{Z}$  now, and  $Z_p = \ker \partial_p, B_p = \text{im } \partial_{p+1}$ . Similarly,  $H_p = Z_p/B_p$ .

Let us consider our popular example now. We orient the triangles CCW, and edges lexicographically.

We now consider 1-cycles in  $K$ . In particular, we consider representative cycles around the hole. Each cycle is also oriented here.



Consider the two cycles  $\bar{c}$  and  $\bar{x}$  shown here which both go CCW around the hole in  $K$ . As vectors, we can write them as follows.

$$\bar{c} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

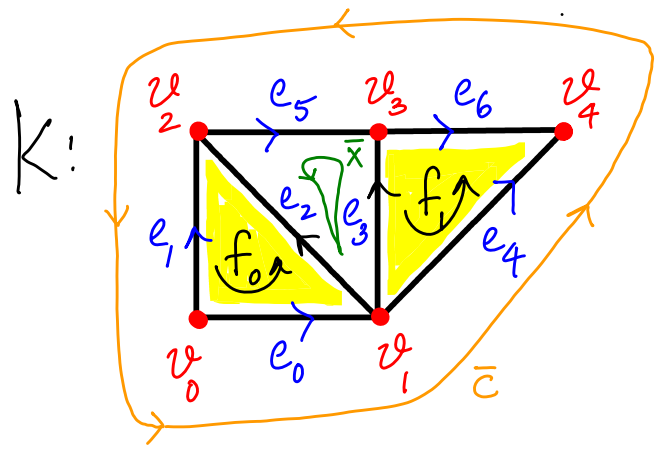
$e_1$  is directed opposite to the orientation of  $\bar{c}$

$$\bar{x} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\bar{c}$  and  $\bar{x}$  are both cycles around the same hole, and hence are homologous. Similar to how we wrote such homology relations over  $\mathbb{Z}_2$  using the boundary matrix, we form the boundary matrix over  $\mathbb{Z}$  here. Now, the boundary matrices have entries in  $\{-1, 0, 1\}$ .

Compare with  $[\partial_p]$  over  $\mathbb{Z}_2$ , which had entries in  $\{0, 1\}$ .

$$[\partial_2] = \begin{matrix} & f_0 & f_1 \\ \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{matrix}$$



In general,  $[\partial_p]$  is an  $m \times n$  matrix when  $K$  has  $m$   $(p-1)$ -simplices  $\tau_i$  and  $n$   $p$ -simplices  $\sigma_j$ . The  $(i, j)$ -entry is nonzero if  $\tau_i \leq \sigma_j$ , and 0 otherwise. This nonzero entry is  $\pm 1$  if orientations of  $\tau_i$  and  $\sigma_j$  agree, and  $-1$  if they are opposite.

Homology between  $\bar{c}$  and  $\bar{x}$  can be written in terms of  $[\partial_2]$ , as follows.

$$\bar{x}: \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \bar{c}: \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \underbrace{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{[\partial_2] \bar{y} \text{ where } \bar{y} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}} \quad \text{So, } \bar{x} \sim \bar{c} \text{ here.}$$

Add boundaries of two triangles  $f_0, f_1$ , after reversing their orientations, to  $\bar{c}$  to get  $\bar{x}$ .

Note that  $\bar{c}$  has 5 edges and  $\bar{x}$  has 3. Even if we were to include (Euclidean) lengths of the edges, and define

$$\|\bar{c}\| = \sum_{i=0}^6 w_i |c_i| \quad \text{where } w_i = 1 \text{ for horizontal/vertical edges } \{0,1,3,5,6\} \\ = \sqrt{2} \text{ for diagonal edges } \{2,4\}$$

We see that  $\|\bar{x}\| = 2 + \sqrt{2}$  and  $\|\bar{c}\| = 4 + \sqrt{2}$ .

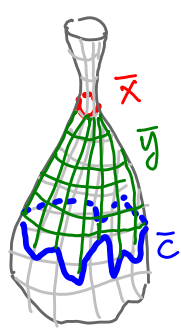
More generally, we could take  $w_i$  as the length of an edge, or the area of a triangle. Irrespective of the direction of traversal of an edge, we want to count its length — hence the absolute value  $|x_i|$ . We can seek a minimal weight chain  $\bar{x}$  that is homologous to  $\bar{c}$ .

### Optimal Homologous Cycle Problem (OHCP) → could generalize to chain when seeking $\bar{x} \sim \bar{c}$ for a chain $\bar{c}$ .

Given  $\bar{c}$  in  $[\bar{c}] \in H_p(K)$ , find  $\bar{x} \in [\bar{c}]$  with  $\|\bar{x}\| = \sum_{i=1}^m w_i |x_i|$ , where  $w_i \geq 0$  is the weight for  $\sigma_i$  the  $i^{\text{th}}$  p-simplex, is smallest.

We can prove an optimal homologous cycle always exists assuming  $K$  is finite, and  $w_i \geq 0 \forall i$ . → simplicial complex

We can cast OHCP as the following optimization problem:



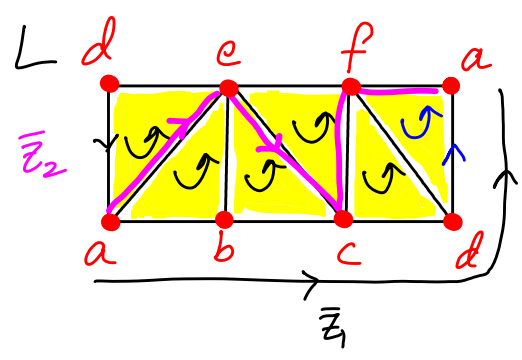
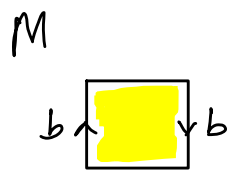
$$\text{minimize } \sum_{i=1}^m w_i |x_i| + \lambda \left( \sum_{j=1}^n v_j |y_j| \right) \\ \text{Subject to } \bar{x} = \bar{c} + [\partial_{ph}] \bar{y} \\ \bar{x} \in \mathbb{Z}^m, \bar{y} \in \mathbb{Z}^n$$

→ generalization where we also minimize the area of triangles used to define  $\bar{x} \sim \bar{c}$ .  $\lambda \geq 0$  is a scale parameter that controls the relative importance of the two terms in the objective function.

Working over  $\mathbb{Z}$  instead of  $\mathbb{Z}_2$ , gives us more "freedom", and actually makes the OHP easier to solve under certain settings!

## Homology over $\mathbb{Z}$ for some familiar objects

### 1. Möbius strip



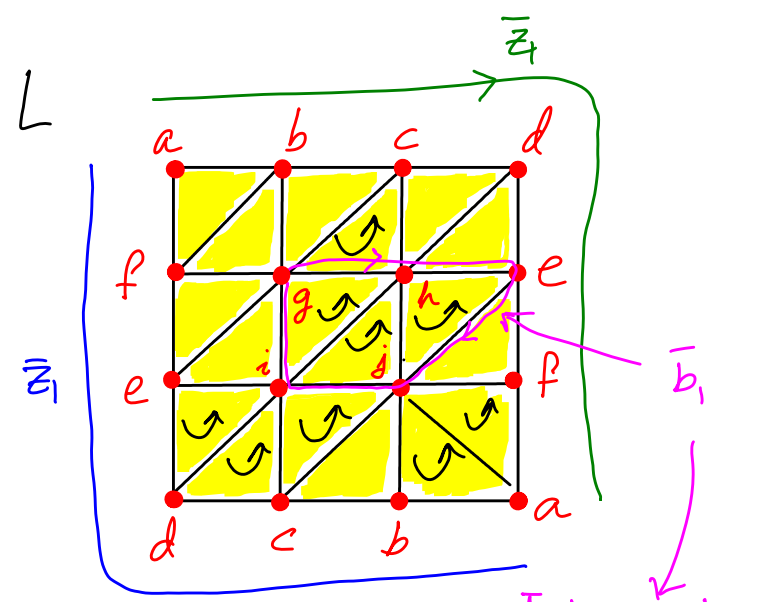
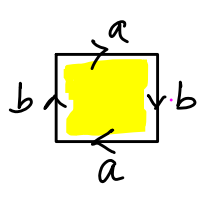
$$\bar{z}_1 = a-b-c-d-a$$

$$\bar{z}_2 = a-e-e-f-a$$

$\bar{z}_1 - \bar{z}_2 = \partial_2 d$ , where  $d$  is the 2-chain of triangles between  $\bar{z}_1$  and  $\bar{z}_2$ .

We could go around  $\bar{z}_1$  any integer # times. Hence  $H_1 \cong \mathbb{Z}$ ,  $\{\bar{z}_1\}$  is a basis, and  $\beta_1(L; \mathbb{Z}) = \text{rank}(H_1; \mathbb{Z}) = 1$  here.  $H_0(L; \mathbb{Z}) \cong \mathbb{Z}$  as well (could use any vertex as a basis). Here,  $\{\beta_0, \beta_1\}$  over  $\mathbb{Z}$  coincide with those over  $\mathbb{Z}_2$ .

### 2. Projective Plane



$H_0(L; \mathbb{Z})$  has same rank as  $H_0(L; \mathbb{Z}_2)$  ( $\beta_0 = 1$  over both  $\mathbb{Z}$  and  $\mathbb{Z}_2$ )

$b_1$  is a 1-boundary bounding the 3 triangles inside.

There is 1 connected component, so  $H_0(K; \mathbb{Z})$  could be expected to be similar to  $H_0(K; \mathbb{Z}_2)$ .

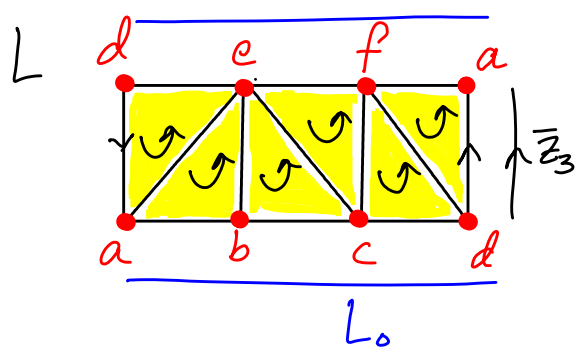
Let's consider  $H_1$ .

We can start by assigning CCW orientations for all triangles. Then the induced orientations for all edges "in the middle" are indeed opposite.

Consider the cycle  $\bar{z}_1 = abcdefa$ . It turns out that  $2\bar{z}_1$  is a boundary — it bounds the 2-chain that has all triangles with coefficient 1

So, we could repeat  $\bar{z}_1$  an odd # times to get another cycle in  $H_1$ . But an even # times gives a boundary. So, here  $H_1 \cong \mathbb{Z}_2$ .

### Back to Example 1: Möbius strip



Let  $L_0 \subset L$  consist of the edges  $\bar{ab}, \bar{bc}, \bar{cd}, \bar{de}, \bar{ef}, \bar{fa}$ , and the vertices. Let  $\bar{z}_3 = \bar{ad}$ .

The induced orientations on  $\bar{ad}$  from  $\triangle ade$  and  $\triangle adf$  do not cancel.

What is  $H_1(L, L_0; \mathbb{Z})$ ?

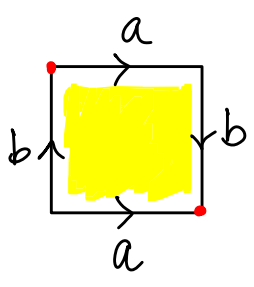
Note that all edges going "across" ( $\bar{ad}, \bar{ae}, \bar{be}, \bar{ce}$ , etc.) are relative 1-cycles. Also, if  $\bar{d}$  is the 2-chain with coefficients 1 for each triangle, then  $\partial_2 \bar{d} = 2\bar{z}_3$ . Thus,  $\bar{z}_3$  is not a relative 1-boundary, but  $2\bar{z}_3$  is.

$$\Rightarrow H_1((L, L_0); \mathbb{Z}) \cong \mathbb{Z}_2.$$

We will see later that Möbius strips are a canonical way in which homology groups that are not  $\cong \mathbb{Z}$  show up in surfaces, and in 2-complexes more generally.

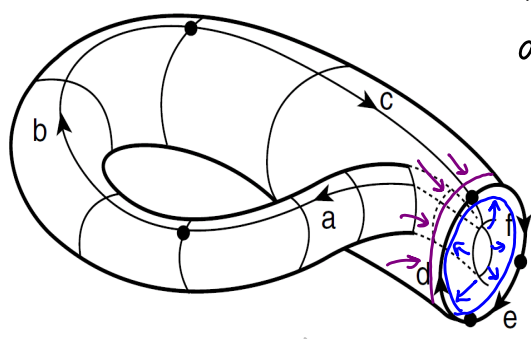
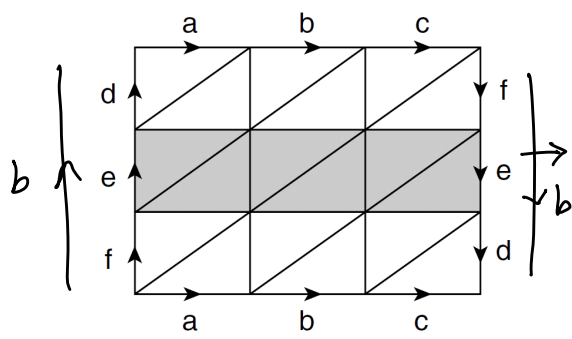
In fact, the "canonical" way in which a 2-manifold (w/ or w/o boundary) is non-orientable is by having a Möbius strip as a subcomplex.

### 3. Klein Bottle ( $\mathbb{K}^2$ )



Notice that gluing along  $b$  produces an "intermediate" Möbius strip (before we also glue along  $a$ ). Going multiple times around  $a$  shows the same behavior —  $a$  essentially behaves like the "skeleton" of this Möbius strip, thus capturing the main "hole" of the strip.

But, going around  $b$  twice after gluing along  $a$  gives a boundary — the boundary of the entire piece of paper. See the alternative triangulation and gluing below.



$2 \times$  forms the boundary of the entire surface, when taken twice — once for the part "outside" and the second for the part "coming through".

(images from P.J. Giblin - "Graphs, Surfaces, & Homology")

Hence  $H_1(\mathbb{K}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$

↙ "free part" of  $H_1$ 
↘ "torsion part" of  $H_1$ 
↘ also written as  $\mathbb{Z}/2$

Notice that going just once around gives a cycle that is not a boundary.