MATH 529: Lecture 28 (04/23/2026)

Today: * total unimodularity (TU)
* OHCP and TU

Recall $B \in \mathbb{Z}^{m \times n}$ is TU if every subdeterminant is $-1, 0,$ or 1.

Examples

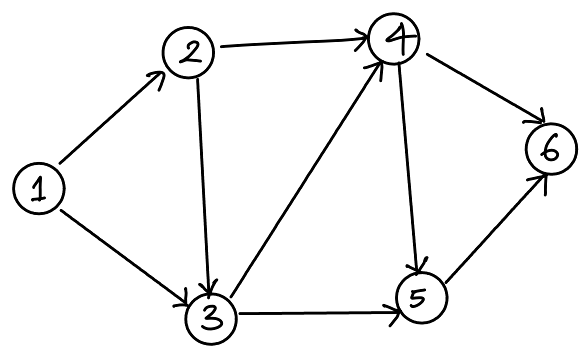
$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is not TU, as $\det B = 2$. But smaller subdeterminants are $0, \pm 1$ in both cases

$B' = \begin{bmatrix} 1 & & -1 \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$ is TU (note that $\det B' = 0$).

We will study these types of matrices in detail soon!

The node-arc incidence matrix of a directed network is TU:

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccccc} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{array} \\ \begin{bmatrix} -1 & -1 & & & & & & & \\ 1 & & -1 & -1 & & & & & \\ & 1 & 1 & & -1 & -1 & & & \\ & & & 1 & 1 & & -1 & -1 & \\ & & & & & 1 & 1 & & -1 \\ & & & & & & & 1 & 1 \end{bmatrix} \end{array}$$



row $\equiv$ node, column $\equiv$ arc

Many network flow problems including min-cost flow, max flow, shortest path, etc, are easy because of this network matrix property. At the same time, they are easier than general LP — there are efficient algorithms to solve them that exploit the network structure.

We consider whether we could use the TU result for OHCP.

When is $A = [I \; -I \; -B \; B]$ TU? → A in the OHCP LP written as $A\bar{x} = \bar{b}, \bar{x} \geq 0$.

Theorem 1 A is TU iff B is TU.

Proof There are several elementary operations that preserve TU.

* taking transpose
* multiply a column/row by -1
* add copy of a row/column
* swap two rows (or two columns)
* add a new singleton row/column with the single non-zero entry being ± 1.
* ...

We could prove these results using arguments that show preservation of determinant (absolute) values under each operation.

We get the constraint matrix A from B using a series of these TU-preserving operations:

$$A = [I \; -I \; -B \; B]$$

* duplicate columns of B
* scale columns of (one copy of) B by -1
* add $2m$ columns of unit vectors

$\square$

Q. When is $B = [\partial_{p+1}]$ TU?

This is the big question now. If B is TU, then we could solve all OHCP instances easily on that K.

Before that, let's revisit OHCP over $\mathbb{Z}_2$.

We could implement homology over $\mathbb{Z}_2$ by modifying the constraints of the OHCP IP as follows.

$$\bar{x}^+ - \bar{x}^- = \bar{c} + B(\bar{y}^+ - \bar{y}^-) + 2\bar{u}, \quad u \in \mathbb{Z}^m.$$

Here, $A = [I \ -I \ -B \ B \ -2I]$, and A is not TU even when B is, because of the $2I$ term.

But we could "simulate" working over $\mathbb{Z}_2$ differently. We could add the constraints $\bar{x}^+, \bar{x}^- \leq \bar{1}_m$, the vector of m ones, and also $\bar{y}^+, \bar{y}^- \leq \bar{1}_n$, the vector of n ones. The modified OHCP LP constraints now become the following.

$$\left.\begin{array}{l} \bar{x}^+ - \bar{x}^- - B\bar{y}^+ + B\bar{y}^- = \bar{c} \\ \bar{x}^+ \qquad\qquad\qquad\quad\ \leq \bar{1} \\ \qquad \bar{x}^- \qquad\qquad\quad\ \leq \bar{1} \\ \qquad\qquad \bar{y}^+ \qquad\ \ \leq \bar{1} \\ \qquad\qquad\qquad \bar{y}^- \ \leq \bar{1} \end{array}\right\} \Rightarrow A' = \begin{bmatrix} I & -I & -B & B \\ I_m & & & \\ & I_m & & \\ & & I_n & \\ & & & I_n \end{bmatrix}.$$

Using arguments similar to ones used in Theorem 1, we get that A' is TU iff B is TU. In the optimal solution, we are now guaranteed to get $x_i, y_j \in \{\pm 1, 0\}$.

We now present the first result characterizing when B is TU.

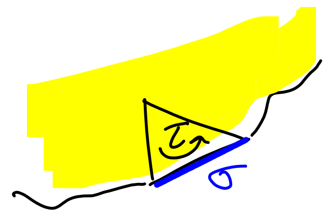
Theorem 2 Let K be a finite simplicial complex triangulating a compact orientable $(p+1)$-manifold. Then $[\partial_{p+1}]$ is TU.

with or without boundary

Proof idea

Case 1: σ is a boundary p-simplex

$$\begin{array}{c|c|} & \tau \\ \sigma & \pm 1 \end{array}$$

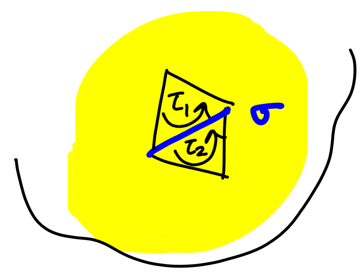


τ : $(p+1)$-simplex
σ : p-simplex, $\sigma \leq \tau$

Row in B corresponding to σ has exactly one nonzero, which is a ± 1 (at column corresponding to τ).

Case 2: σ is a "manifold" p-simplex.

Assume K is consistently oriented.



Here, the row corresponding to σ has exactly two nonzeros, at columns corresponding to τ_1 and τ_2, and these entries are a $+1$ and a -1.

$$\begin{array}{c|cc|} & \tau_1 & \tau_2 \\ \sigma & +1 & -1 \end{array}$$

$\sigma \leq \tau_1$ and $\sigma \leq \tau_2$ here.

To obtain a consistent orientation, we might have to scale some columns (τ_j) by -1, but those operations preserve TU.

So, every row of $[\partial_{p+1}]$ has 1 or 2 non-zeros. If it has two nonzeros, they are $+1$ and -1.

Now, consider any $r \times r$ submatrix S of $[\partial_{p+1}]$.

Rows of S:
* could be all zero
* could have a single ± 1
* have one $+1$ and one -1.

If S has a zero row, then $\det(S) = 0$. If there is a singleton row, we could expand along that row, and look at an $(r-1) \times (r-1)$ subdeterminant instead. In the nontrivial case, every row has one $+1$, one -1.

So assume every row of S has two nonzeros: $+1, -1$.

$\Rightarrow S\bar{1} = \bar{0}$ (adding all columns gives zero vector!)

$\Rightarrow \det S = 0$ (columns are linearly dependent)

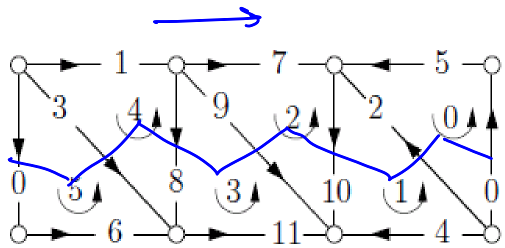
$\Rightarrow B$ is TU.

If K is not consistently oriented to start with, then we multiply a subset of columns of B by -1 to orient K consistently. These scaling operations preserve TU.

$\Rightarrow [\partial_{p+1}(K)]$ is TU for orientable manifold K (with or without boundary). $\square$

What about $[\partial_{p+1}]$ of arbitrary simplicial complexes, that are not necessarily orientable manifolds? We consider perhaps the quintessential nonorientable manifold first — the Möbius strip.

Illustration on Möbius strip



Notice that we have a minimal Möbius strip here — if we remove one triangle, we get a disc, and the Möbius strip disappears.

Hence, to possibly find an obstruction to TU, we look at a submatrix that uses the entire Möbius strip, i.e., all triangles, and hence all columns.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\
S = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} 0 \\ 3 \\ 8 \\ 9 \\ 10 \\ 2 \end{array}
\end{array}
$$

Möbius cycle matrix (MCM)

$\det S = -2$

$[\partial_2]$ for Möbius strip :

$$
\begin{bmatrix}
 & 0: & 1: & 2: & 3: & 4: & 5: \\
0: & 1 & 0 & 0 & 0 & 0 & 1 \\
1: & 0 & 0 & 0 & 0 & -1 & 0 \\
2: & -1 & 1 & 0 & 0 & 0 & 0 \\
3: & 0 & 0 & 0 & 0 & 1 & -1 \\
4: & 0 & -1 & 0 & 0 & 0 & 0 \\
5: & 1 & 0 & 0 & 0 & 0 & 0 \\
6: & 0 & 0 & 0 & 0 & 0 & 1 \\
7: & 0 & 0 & -1 & 0 & 0 & 0 \\
8: & 0 & 0 & 0 & 1 & -1 & 0 \\
9: & 0 & 0 & 1 & -1 & 0 & 0 \\
10: & 0 & 1 & -1 & 0 & 0 & 0 \\
11: & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

Similarly, the boundary edges, i.e., the edges that are faces of only one triangle each, cannot contribute in a nontrivial manner to any determinants. So, we take all the "manifold" edges shared by the 6 triangles to consider the 6×6 submatrix using rows 0, 2, 3, 8, 9, 10, and all the columns 0–5. Indeed, this submatrix has determinant -2. Furthermore, if we rearrange the rows and columns in the order in which we see the edges and triangles from left to right, we see a canonical matrix, which we call **Möbius cycle matrix** (MCM).

$$\mathrm{MCM}(n) = \begin{bmatrix} 1 & & & & \alpha \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}, \quad \alpha = (-1)^{n+1}. \qquad \det(\mathrm{MCM}(n)) = 2.$$

We can prove that in 2D (i.e., for the edge-triangle case), the only obstructions to $[\partial_2]$ being TU are these Möbius strips. The corresponding subcomplex is called a Möbius subcomplex.

We can show that the only way in which a 2-complex has a non-TU boundary matrix is by it having a Mobius subcomplex.

Theorem 3 $[\partial_2(K)]$ is TU iff K has no Möbius subcomplex.

Proof uses result on minimal violators of TU of matrices. These are matrices that are not TU, but every proper submatrix is TU. These minimal violators belong to two classes. One is the MCMs. In the second class, every row/column has at least 4 non-zeros. For a 2-complex, every column has exactly 3 nonzeros, and hence cannot have a submatrix of the second class. $\square$