MATH 529: Lecture 2 (01/15/2026)

Today: * topology, open/closed sets
* homeomorphism, examples

We define topology as a mathematical method to define and study how a space is connected.

Notation For a set X, we denote by 2^X the power set of X, which is the set of all subsets of X.

Def A **topology** on a set X is a subset T of 2^X such that the following conditions hold.

1. $A_1, A_2 \in T \Rightarrow A_1 \cap A_2 \in T$ (finite intersections)

2. $\{A_j \mid j \in J\} \in T \Rightarrow \bigcup_{j \in J} A_j \in T$ (infinite unions or finite)

index set infinite or finite

3. $\phi, X \in T$ empty set

(X, T) is a topological space, denoted $\mathbb{X}$ (T is understood from context).

$A \in T$ is an **open set** of $\mathbb{X}$.

The complement of A, i.e., $X - A$ (or $X \setminus A$) "set minus" is a **closed set**.

Some sets can be both open and closed at the same time, e.g., ϕ, X are both open and closed in any topology.

We typically specify a topology by specifying its open sets.

interior $\mathring{A}$ of $A \subseteq \mathbb{X}$: $\mathring{A} = \bigcup$ (open sets contained in A)

union

closure $\bar{A}$ of $A \subseteq \mathbb{X}$: $\bar{A} = \bigcap$ (closed sets containing A).

intersection

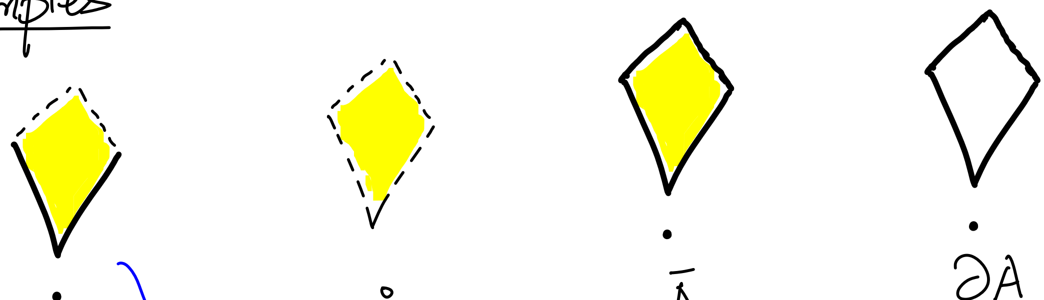
minimal closed set that contains A.

boundary ∂A of $A \subseteq \mathbb{X}$: $\partial A = \bar{A} - \mathring{A}$.

$\partial A = \{$points in A that intersect both $\bar{A}$ and $\overline{(X-A)}\}$.

Examples

1.



$A \subseteq \mathbb{X}$ $\quad$ $\mathring{A}$ $\quad$ $\bar{A}$ $\quad$ ∂A

A consists of the "half-open" rhombus and a separate point.

2. A discrete example. Let $X = \{a, b, c\}$.

We can define different topologies on X.

Let $T_1 = \{\phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and

$T_2 = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

Under T_1, $\{a,b\}$ is open, its complement $\{c\}$ is closed. With $A = \{a,b\}$, $\mathring{A} = \bigcup\{\phi, \{b\}, \{a,b\}\} = \{a,b\} = A$.

We can specify other topologies on X, e.g., $T_3 = 2^X$, where each set in T is both open and closed. But $T_4 = \{\phi, \{a\}, \{b\}, \{a,b,c\}\}$ is not a topology, as, e.g., $\{a\} \cup \{b\} = \{a,b\} \notin T_4$.

Neighborhood Let $\mathbb{X}=(X,T)$. A neighborhood of $x \in X$ is any $A \in T$ such that $x \in \mathring{A}$.

More generally, some books define a neighborhood as any set that includes, i.e., contains as a subset, an open set which contains X. Under this definition, the neighborhood could be a closed set (or neither open nor closed).

Now that we have defined topology, we consider the natural next question of comparing two spaces — how do we say two given spaces have the "same topology"? We introduce the notion of homeomorphism as a (strong) notion of topological similarity.

Homeomorphism

In geometry, we can study transformations that preserve "shape" of a rigid body, e.g., rotation and translation. These transformations "do not change" the geometry of the body.

In topology, we permit more types of transformations — e.g., stretch, shrink, expand, twist, etc., as long as you do not cut one piece into two or more, or join two pieces into one, or poke a hole in your object. All such permitted transformations "preserve topology".

A series of such permitted transformations that preserve topology constitute a homeomorphism. And two spaces are topologically "similar" if such "nice" functions exist from one space to the other and also back. We define what we mean by "nice" here.

We start with some background and definitions on functions.

Def Let A, B be sets. A function $f: A \to B$ is a rule that assigns exactly one $b \in B$ for every $a \in A$.

dom f: domain of $f = A$, cod f: codomain of $f = B$

im f: image of $f = \{b \in B \mid f(a) = b \text{ for some } a \in A\} = \{f(a) \mid a \in A\}$.

im f is also called the range of f. Note that im $f \subseteq$ cod f.

"for every"

$f: A \to B$ is 1-to-1 or **injective** if $\forall b \in B$, there exists **at most** one $a \in A$ with $f(a) = b$.

can be none

$f: A \to B$ is onto or **surjective** if $\forall b \in B$, there exists **at least** one $a \in A$ with $f(a) = b$.

can be more

If f is both injective and surjective, we say that f is **bijective**, or that f is a bijection.

Def A function $f: \mathbb{X} \to \mathbb{Y}$ is **continuous** if for every open set $B \subseteq \mathbb{Y}$, $f^{-1}(B)$ is open in $\mathbb{X}$. "takes" open sets to open sets.

A continuous function is also called a **map**.

Def A **homeomorphism** $f: \mathbb{X} \to \mathbb{Y}$ is a bijective function such that both f and f^{-1} are continuous.

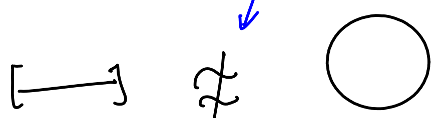
We say $\mathbb{X}$ is homeomorphic to $\mathbb{Y}$, or $\mathbb{X} \approx \mathbb{Y}$.

We also say that $\mathbb{X}$ and $\mathbb{Y}$ have the same topological type.

Examples

It's often easier to argue why two spaces are not homeomorphic — we just identify one (or more) place(s) where things don't work.

"not homeomorphic"

1. 

$[0,1] \not\approx S^1$

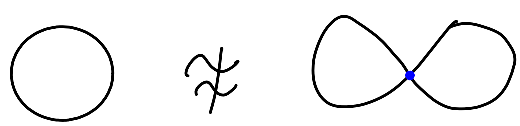
0 1

$[0,1]$ S^1

1-sphere, or circle

We would need a map that assigns both end points of $[0,1]$ to a single point in S^1.

But the inverse of any map that takes both end points of $[0,1]$ to one point in S^1 is not bijective.

2. 

$S^1 \not\approx$ Figure 8

The crossing point in ∞ ($\times$) cannot be mapped to a corresponding point in S^1.

Also, we could map S^1 to one of the two circles in figure-8, but not both.

On the other hand, two show that two spaces are homeomorphic, we need to specify the maps f and f^{-1}!

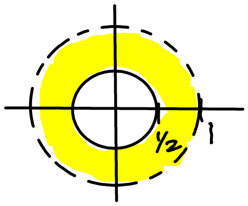
3. 

$$D_1 \underset{f^{-1}=g}{\overset{f}{\rightleftarrows}} \mathbb{R}^2, \quad D_1 \approx \mathbb{R}^2$$

g for ease of notation

$D_1 = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1\} \rightarrow$ open unit disc

Intuitively, we can shrink all of $\mathbb{R}^2$ into D_1. Similarly, we can stretch D_1 to fill all of $\mathbb{R}^2$.



$$g(\bar{x}) = \frac{\bar{x}}{1+\|\bar{x}\|_2} \qquad g : \mathbb{R}^2 \to D$$

Euclidean norm

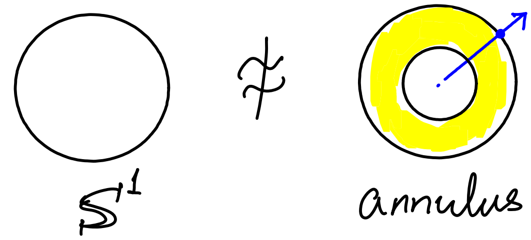
g maps all of D_1 (in $\mathbb{R}^2$) to fit within $D_{1/2} = \{\bar{x} \in \mathbb{R} \mid \|\bar{x}\| < \frac{1}{2}\}$, and then fits all of $\mathbb{R}^2$ outside D_1 within the half open annulus with radii $\frac{1}{2}$ and 1.

The continuous function going from D_1 to $\mathbb{R}^2$ can be similarly defined:

$f : D \to \mathbb{R}^2$ where $f(\bar{x}) = \dfrac{\bar{x}}{1-\|\bar{x}\|}$. f is an "infinite stretch".

Note that points $\bar{x}$ in D that are close to the edge, i.e., have $\|\bar{x}\|$ close to 1, are mapped so as to fill up the entire $\mathbb{R}^2$ outside $\bar{D}$. We stretch the open disc so as to fill the entire plane, and hence it is called an infinite stretch.

Usually, we try to define the continuous maps f and f^{-1} to show that two spaces $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic. At the same time, the intuition (geometric when possible) is also important to grasp. On the other hand, to show that $\mathbb{X} \not\approx \mathbb{Y}$, it is often sufficient to identify subset(s) that create the obstructions, e.g., the $\times$ in figure-8 v/s $\mathbb{S}^1$.

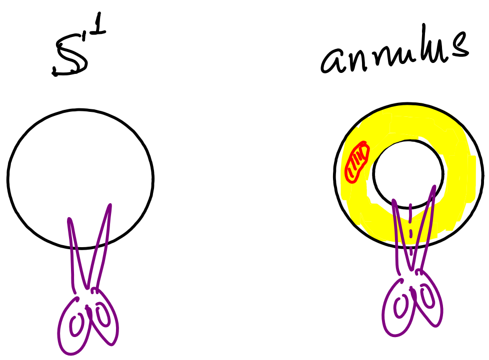
4. 

Both these spaces have the shape of a "hole".

We could shrink the annulus so that it reduces to the circle. The corresponding function maps every point on the annulus radially onto the outer circle, for instance. But we cannot uniquely map the circle back to the annulus — would need to "map" each point on the circle to (infinitely) many points on the thick strip of the annulus.

Another observation highlights the neighborhoods of points in the circle and the annulus. Every point on the circle has open neighborhoods that look like the number line ($\mathbb{R}^1$). On the other hand, points in the annulus have neighborhoods that look like the open disc ($\mathbb{R}^2$) or open half disc (the points on the boundary).

Intuitively, the annulus is 2-dimensional, while S^1 is one-dimensional.



Notice that the two spaces behave the same way under a "cut" as we had been talking about earlier with the string.

In particular, a straight cut along one "edge" of either space would leave them both connected. At the same time, one could "carve out" a 2D disc ( ) from the annulus, but not from the circle.

If we "relax" our definition of topological similarity, the two spaces would be considered the same — they both look like a hole, after all. Indeed we will see that checking for homeomorphism is difficult (both theoretically and computationally). We'll work with looser concepts of topological similarity later on — homology!