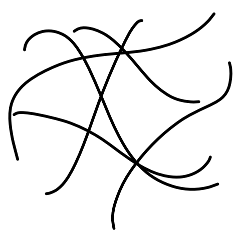
MATH 529: Lecture 30 (04/30/2026)

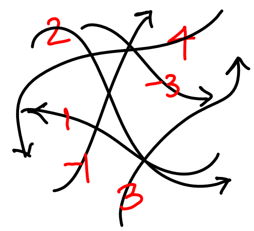
Today: * GMT : currents and flat norm
* flat norm as an OHCP

Connection to Geometric Measure Theory (GMT)

Currents: generalized surfaces used in GMT



A "rectifiable" 1-set



1-current

Orient each piece, and assign an integer multiplicity

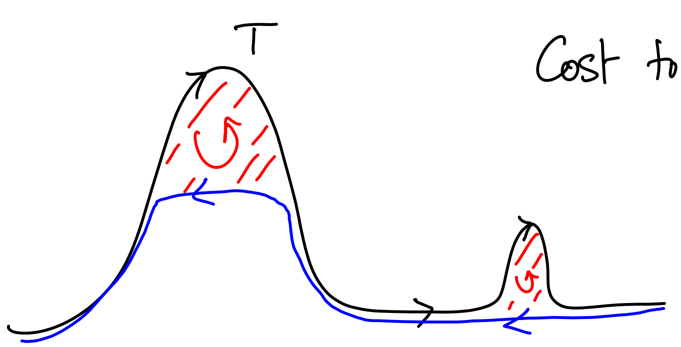
In simplicial setting, we study 1-chains (over $\mathbb{Z}$).

Flat norm of d-current T with scale ($\lambda \geq 0$)

$$\mathbb{F}(T) = \min_{S} \{ \underbrace{M_d}_{\downarrow}(T - \partial S) + \lambda M_{d+1}(S) \}$$

T: d-current
S: (d+1)-current

Mass (d-dim volume ≡ length, area, volume etc.)



Cost to erase T using two operations:

Operation 1: draw 1-curves of opposite orientation
cost: length of curve

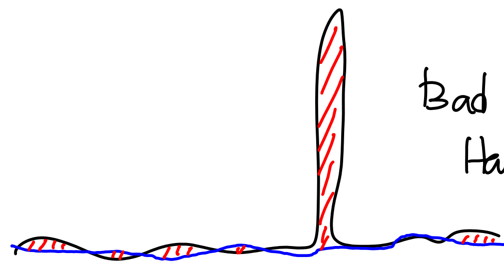
Operation 2: trace boundaries of 2D patches
cost: area of patch

Total cost = length + area

Flat norm (T) = min total cost over all such decompositions.

Distance between two currents P and Q: $\text{dist}(P,Q) = \mathbb{F}(P-Q)$.

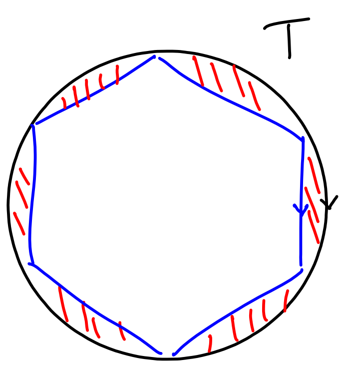
The flat norm distance is the "natural" distance between spaces modeled as currents. Hausdorff distance is sensitive to outliers. Consider the two curves (one blue and another black), which are close to each other except for a small narrow spike in the black curve. Hausdorff distance between the curves will be offset by this spike. But the flat norm distance will be small — the area of the spike in essence.



Bad for Hausdorff

Another option is the Frechét distance, or "dog leash" distance. Imagine you're walking on the blue curve and your dog on the black curve. Both could stop or move at different speeds, but cannot go backwards. The Frechét distance is the Length of the smallest (non-stretching) leash that'll work. But this distance is not defined for more general inputs, e.g., the spagetti goop!

What about the mass difference $M(P-Q)$? Even this distance might not make sense in all cases, though. Consider the following example.



T: unit circle (oriented CW → clockwise)

T_n: regular n-gon inscribed in T

Notice that $M(T-T_n) \to 4\pi$ as $n \to \infty$.

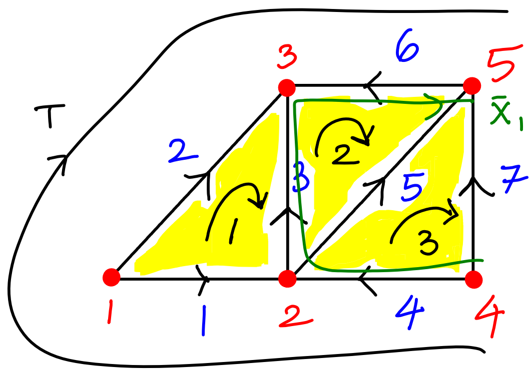
But $\mathbb{F}(T-T_n) \to 0$, as the flat norm distance between T and T_n is measured by the area between T and T_n, which goes to zero as $n \to \infty$.

Simplicial Flat Norm

We can represent currents as chains on simplicial complexes, and can compute the simplicial flat norm as an optimal homology problem.

Example

$$\bar{t} = \begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array}\begin{bmatrix} -1\\ 1\\ 0\\ 1\\ 0\\ -1\\ 0 \end{bmatrix}$$



Note that length$(T) = 3+\sqrt{2}$
$= 4.41$

$$w_i = \begin{cases} 1, & i = 1,3,4,6,7 \\ \sqrt{2}, & i = 2,5 \end{cases}$$

For $\lambda = 2.5$ we get

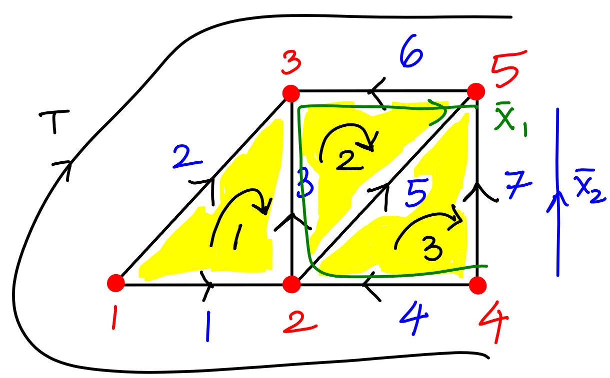
$$F_\lambda(\bar{t}) = 3 + \left(\tfrac{5}{2}\right)\left(\tfrac{1}{2}\right) = 4.25$$

$$< \text{length}(\bar{t}) = 4.41$$

Let $v_j = \frac{1}{2}\ \forall j$. (area of each triangle)

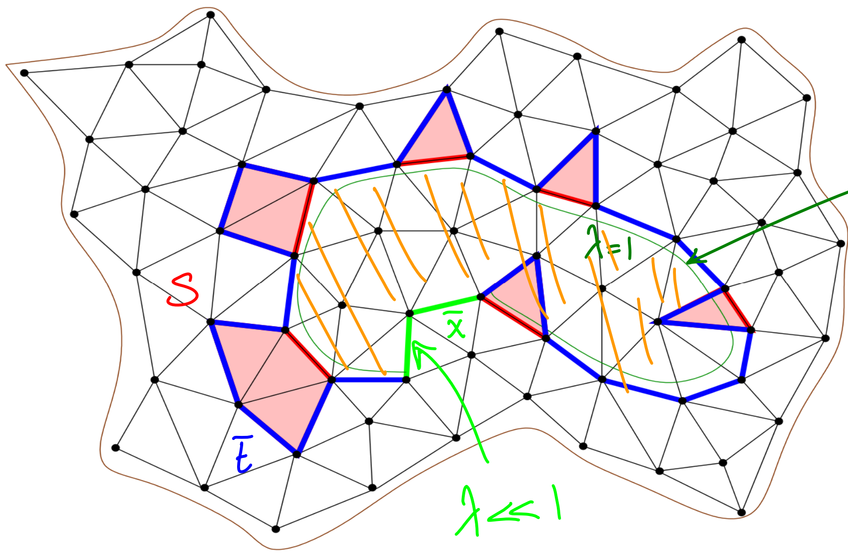
For $\lambda = 1$, we get $\bar{x}_2$ as the result;

$$F_\lambda(\bar{t}) = 1 + \overset{\lambda}{(1)}\underbrace{\left(3\left(\tfrac{1}{2}\right)\right)}_{\text{area of 3 triangles}} = 2.5$$



$\bar{x}_2$ remains the flat norm decomposition for any $\lambda \leq 1$.

λ is a scale parameter. At smaller values of λ, we smooth out larger scale features.



flat norm decomposition for λ=1 (using S)

For λ<<1, we use almost all the 2D space "enclosed" by E.

Simplicial Flat Norm as an LP

We could write down an integer program for the simplicial flat norm problem that is quite similar to the OHCP IP. We start with the optimization model with absolute value terms.

$$\min \sum_{i=1}^{m} w_i |x_i| + \lambda \left(\sum_{j=1}^{n} v_j |s_j| \right)$$

piecewise linear

$$\bar{x} = \bar{t} - [\partial_{p+1}]\bar{s}$$

$$\bar{x} \in \mathbb{Z}^m, \quad \bar{s} \in \mathbb{Z}^n$$

Using the standard transformation to handle absolute values, we get the following IP:

$$\min \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \lambda \sum_{j=1}^{n} v_j (s_j^+ + s_j^-)$$

recall that we had used $\bar{y}^+, \bar{y}^-$ in place of $\bar{s}^+, \bar{s}^-$ previously

$$\text{s.t.} \quad \bar{x}^+ - \bar{x}^- = \bar{t} - [\partial_{p+1}](\bar{s}^+ - \bar{s}^-)$$

$$\bar{x}^+, \bar{x}^- \geq \bar{0}, \quad \bar{s}^+, \bar{s}^- \geq \bar{0}$$

$$\bar{x}^+, \bar{x}^- \in \mathbb{Z}^m, \quad \bar{s}^+, \bar{s}^- \in \mathbb{Z}^n$$

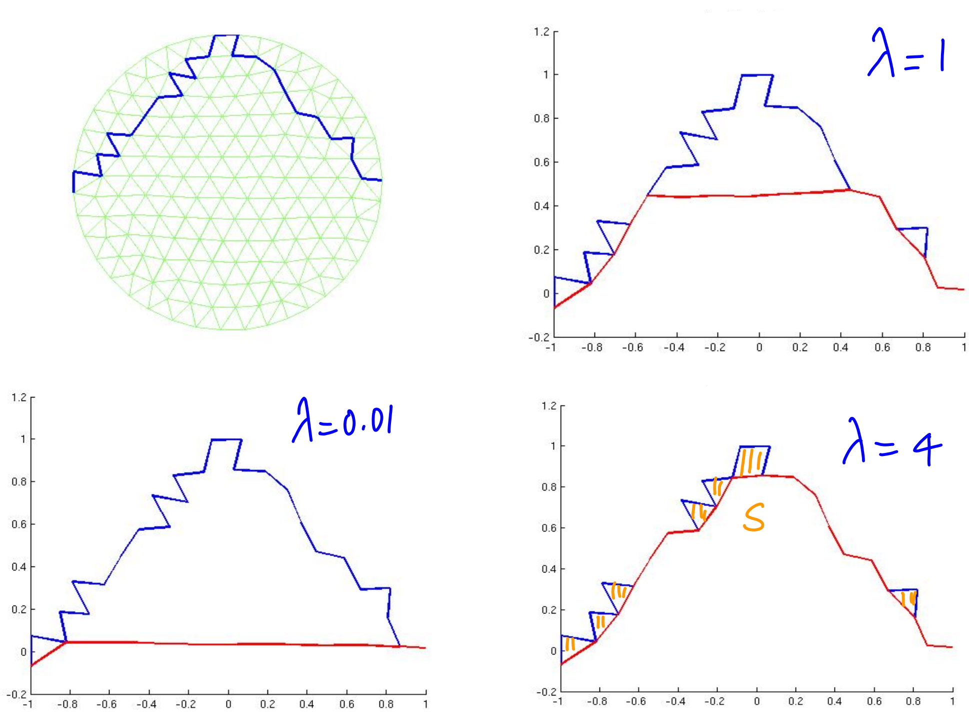
relax to get MSFN LP

This is the Multiscale simplicial flat norm (MSFN) LP. Note that we can use results on TU of $[\partial_{p+1}]$ here, just as we did for the OHCP.

When $\lambda = 0$, the MSFN LP $\equiv$ OHCP LP.

When $\lambda \to \infty$, the MSFN LP solves the area minimizing surface problem for a cycle $\bar{E}$.

Here is an illustration in 2D (for 1-chain $\bar{E}$). While the definition allows one to consider all of $\mathbb{R}^2$ as candidates for S, we could restrict to the "convex hull" of E.



At larger values of $\lambda = 4$, we "smooth out" only the small scale bumps in $\bar{E}$. At $\lambda = 0.01$, we flatten out the entire input curve.

It appears the S chains seem to only grow as λ decreases.

Open problem: When can you guarantee that as $\lambda \to 0$, the $(d+1)$-chains S (defining $\bar{E} \sim \bar{x}$ homology) form a filtration?

Could we devise an "incremental algorithm" for computing the flat norm?