MATH 529: Lecture 4 (01/22/2026)

Today: * d-manifolds in general
* Classification of 2-manifolds

We will now introduce some concepts which we will use to define d-manifolds in general (in particular, for $d \geq 2$).

Def A **cover** of $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X such that $A \subseteq \bigcup_{j \in J} C_j$.
↳ index set

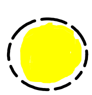
An **open cover** is a cover made of open sets.
index set is a subset

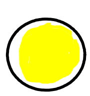
A **subcover** of A is a cover $\{C_k \mid k \in K\}$ such that $K \subseteq J$.

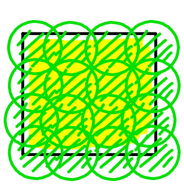
Def A set A is **compact** if every open cover of A has a finite subcover. Correspondingly, a topological space $\mathbb{A} \subseteq \mathbb{X}$ is compact if every open cover of $\mathbb{A}$ has a finite subcover.

Note: In $\mathbb{R}^d$, closed + bounded $\Longleftrightarrow$ compact.

e.g., S^2 is compact, but $\mathbb{R}^2$ is not.

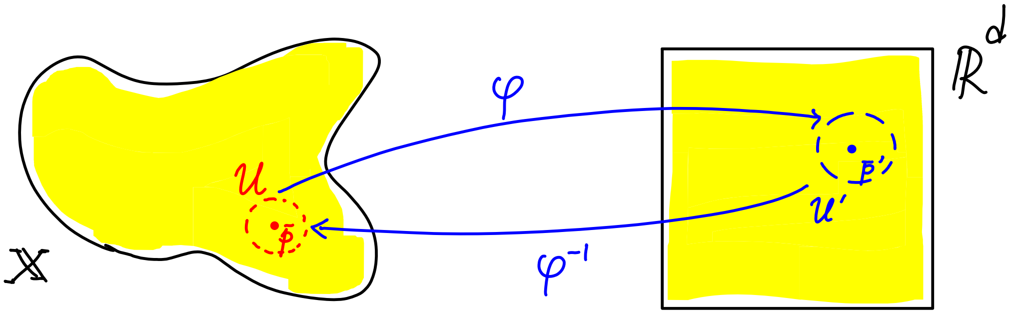
 D_1 is not compact.

 $\overline{D}_1$ (closed disc) is a compact 2-manifold with boundary



Example of finite subcover: consider a unit square, which is a subset of $\mathbb{R}^2$. Consider open discs of radius $1/4$ centered at each rational point within the square (denoted here as ). There are infinitely many such discs, which together cover the square. But a finite subset of those discs also covers the square.

Def A **chart** at $p \in \mathbb{X}$ is a homeomorphism $\varphi : \mathcal{U} \to \mathbb{R}^d$ for $\mathcal{U} \in \mathbb{X}$ an open set containing p. The **dimension** of the chart is d.



Def (Hausdorff) A topological space $\mathbb{X}$ is **Hausdorff** if $\forall x, y \in X$, $x \neq y$, there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of x, y, respectively, such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.

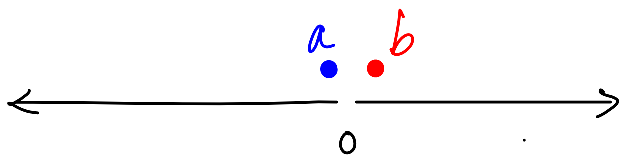
an open set containing the point in question

e.g., $\mathbb{R}^2$.

Example of a non-Hausdorff space: $X = L \cup \{a, b\}$ where

$L = \mathbb{R} \setminus \{0\}$

a and b are both used in place of the origin. Open sets are the usual open intervals in $\mathbb{R}$.



and such that

$L \cup \{a\} \approx \mathbb{R}$ and $L \cup \{b\} \approx \mathbb{R}$.

But every pair of open sets $\mathcal{U}$ and $\mathcal{V}$ containing a and b, respectively, intersect!

Def A topological space is **completely separable** (or second countable) if it has a countable basis, i.e., it has a countable collection of open sets such that every open set can be written as a union of open sets from this collection (basis).

think $\mathbb{Z}$, integers, as opposed to $\mathbb{R}$, which is uncountable

e.g., $\mathbb{R}$ is completely separable – it can be shown that open intervals with rational lengths centered at only rational points works as a countable basis.

A space that is not completely separable: take uncountably many copies of $[0,1)$, e.g., with the 0 of $[0,1)$ anchored at all irrational points – called the long line or Alexandroff line.

Def (Manifold) d-dimensional manifold A completely separable, Hausdorff space $\mathbb{X}$ is a **d-manifold** if there exists a d-dimensional chart at every $\bar{x} \in \mathbb{X}$, i.e., $\bar{x}$ has a neighborhood homeomorphic to $\mathbb{R}^d$.

$\mathbb{X}$ is a d-manifold with boundary if every $\bar{x} \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}^d = \{\bar{x} \in \mathbb{R}^d \mid x_1 \geq 0\}$ (d-dimensional half space).

The boundary of $\mathbb{X}$, denoted by $\partial\mathbb{X}$, is the set of $\bar{x} \in \mathbb{X}$ with a neighborhood homeomorphic to $\mathbb{H}^d$.

The **dimension** of the manifold is d here.

Notice the correspondence between the definition of 2-manifolds introduced previously, and the general definition here. The main condition is the existence of neighborhoods $\approx \mathbb{R}^d$ around each point.

Def (Embedding) An **embedding** of $\mathbb{X}$ in $\mathbb{Y}$ is a map $g: \mathbb{X} \to \mathbb{Y}$ whose restriction to $g(\mathbb{X})$ is a homeomorphism.

Manifolds are manifolds irrespective of their embedding! $\mathbb{S}^2$ is a 2-manifold even if it is not sitting in $\mathbb{R}^3$.

We will introduce alternative representations of manifolds to highlight this point. In fact, in many cases, we can study the manifold easily using such representations.

Classification of Manifolds (continued)

Enumerate all possible manifolds of a given dimension upto homeomorphism. We already listed the classifications for 0- and 1-dimensional manifolds.

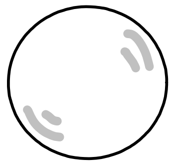
We now consider the case of compact, connected, closed 2-manifolds. We first list the "basic building blocks" so to speak, which include the 2-sphere, torus, Möbius strip, and the real projective plane. We can build larger 2-manifolds by gluing these building blocks together.

2-Manifolds (We consider compact 2-manifolds)

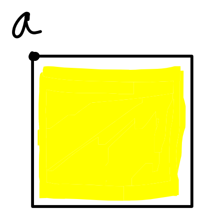
First, let us study several typical 2-manifolds, some of which we already saw previously.

1. $\mathbb{S}^2$

2-sphere



$\{\bar{x} \in \mathbb{R}^3 \mid \|\bar{x}\|_2 = 1\}$

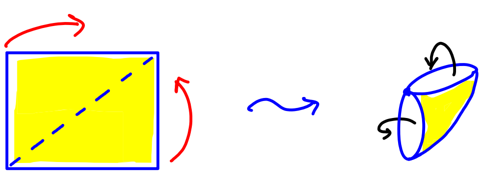


"identify" all points on boundary with the point a.

This is a "diagram" of $\mathbb{S}^2$.

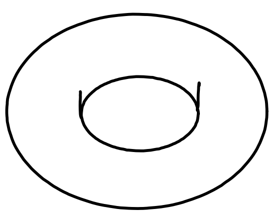
both are 2-spheres!

Start with a square sheet of paper and glue its all its edges together to make a sphere.



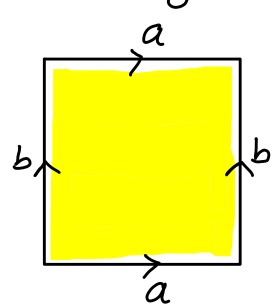
Arrows capture how edges are glued — with or without twist.

2.



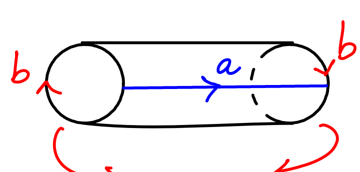
$\mathbb{T}^2$ torus

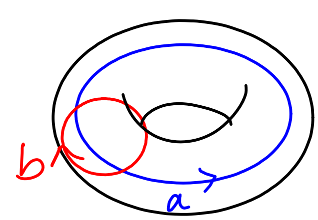
$\approx$



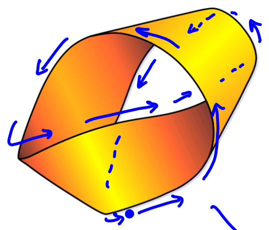
we could mathematically study this representation

Imagine folding a rectangular sheet of paper first into an open cylinder, and then gluing its end circles to form a torus.



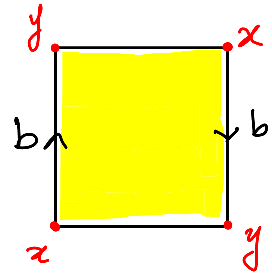


3. Möbius strip → 2-manifold with boundary



(image: www)

$\approx$

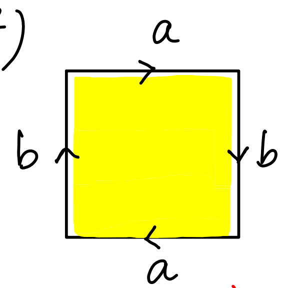


Notice that we can traverse the "edge" of the Möbius strip in one go — it's one big circle!

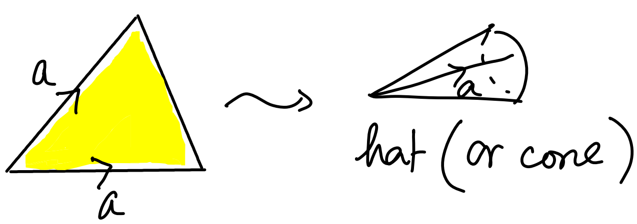
Notice that we are not identifying the horizontal edges. So they remain as boundaries. At the same time all four edges are identified pairwise in the case of the torus. Indeed, the Möbius strip is a manifold with boundary, while the torus is a manifold (without boundary).

cannot be embedded in $\mathbb{R}^3$!

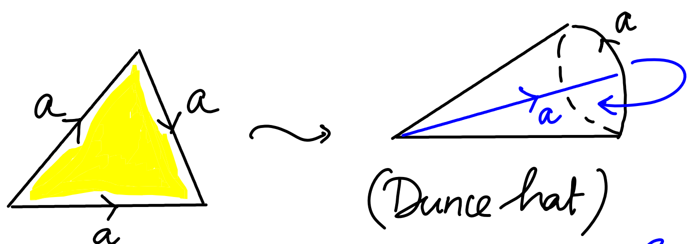
4. (Real) Projective plane ($\mathbb{R}P^2$) (also, Dunce hat)



identify the free edges of Möbius strip in an opposing sense.



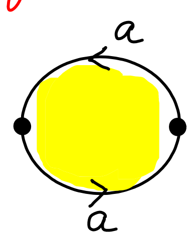
hat (or cone)



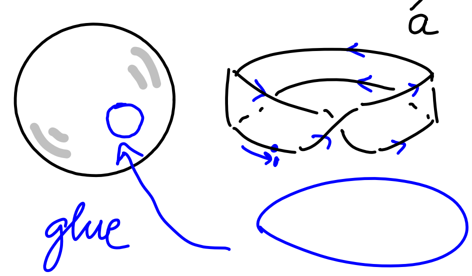
(Dunce hat)

Same as gluing the boundary of a disc to the boundary of a Möbius strip.

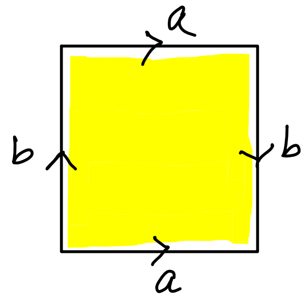
Here's another representation



Yet another way to make $\mathbb{R}P^2$: Cut an open disc out of S^2 and glue a Möbius strip along the edge left by the cut.

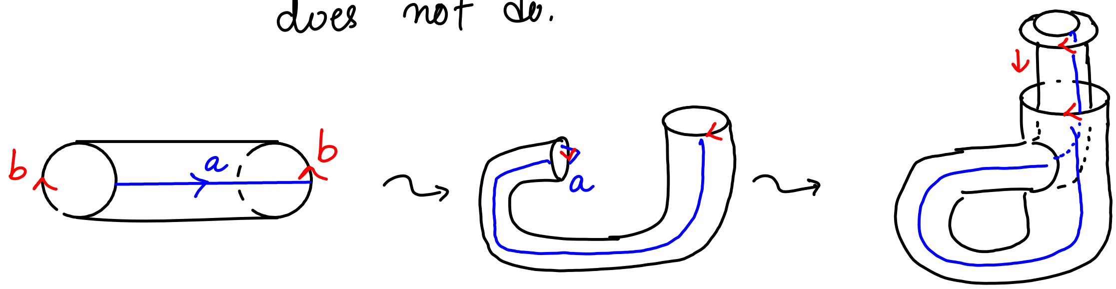


5. Klein bottle ($\mathbb{K}^2$)



Identify free edges of the Möbius strip in the same direction.

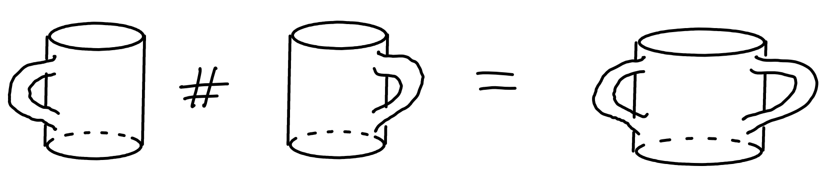
An "immersion" in 3D:
→ allows self intersection, which an embedding does not do.



We get $\mathbb{K}^2$ also by gluing together two Möbius strips along their boundary circles. Or, cut two discs out of $\mathbb{S}^2$ and glue a Möbius strip each along the edges of both cuts.

Note that $\mathbb{S}^2$ and $\mathbb{T}^2$ are orientable manifolds, while the Möbius strip, $\mathbb{RP}^2$, and $\mathbb{K}^2$ are non-orientable manifolds (with or without boundary).

We can obtain more general 2-manifolds by "gluing" these basic shapes together. For example, we can connect two coffee cups to get one coffee cup with two handles!



We can do this kind of "gluing" to join manifolds in any dimension (as long as the manifolds being joined have the same dimension). This ing is formally termed connected sum.

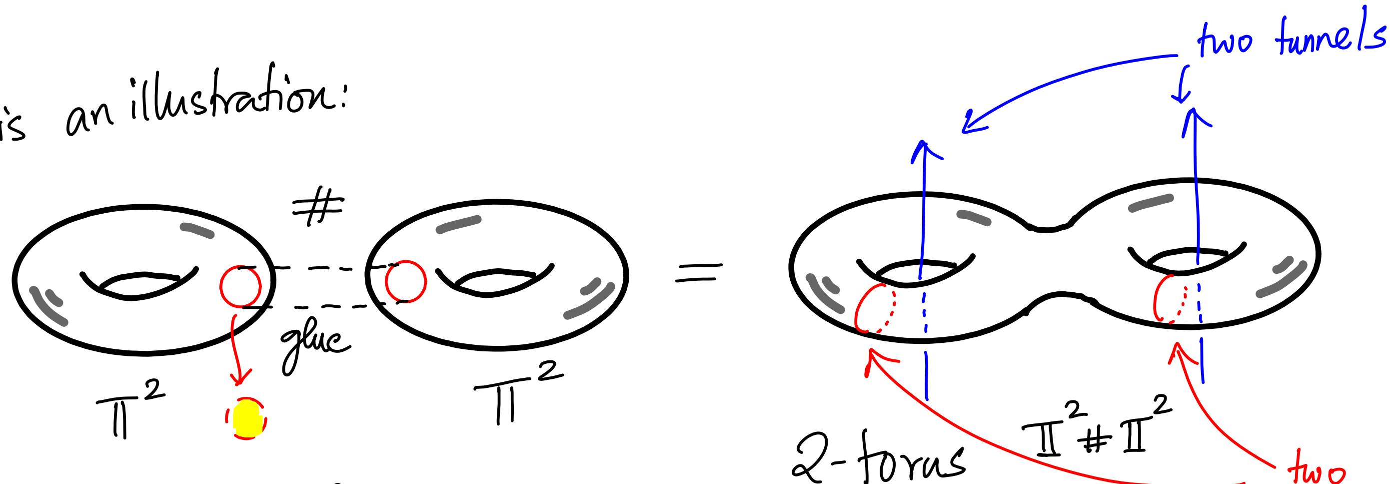
Def (Connected sum). Let M_1, M_2 be d-manifolds. The **connected sum** of these d-manifolds is another d-manifold defined as follows.

$$M_1 \# M_2 = (M_1 - D_1^d) \bigcup_{\partial \bar{D}_1^d \cong \partial \bar{D}_2^d} (M_2 - D_2^d)$$

↳ identified by homeomorphism

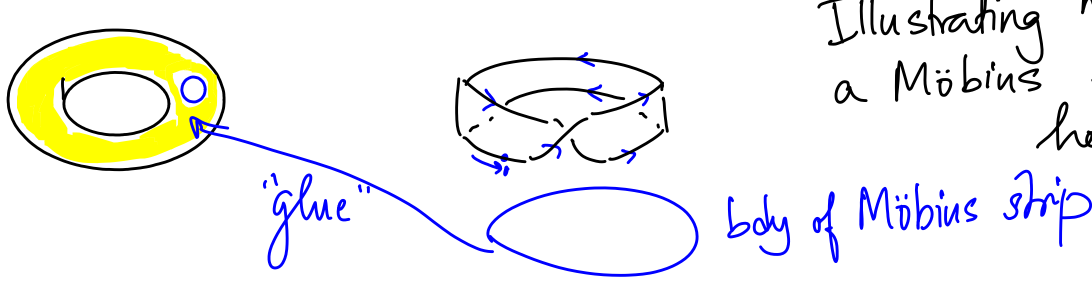
D_1^d, D_2^d are d-dimensional open discs in M_1, M_2 respectively.

Here is an illustration:



Remove open dics from both tori, and "glue" along the boundaries of these circular holes.

2-torus
(double torus)



Illustrating how to glue a Möbius strip to a hole in a torus.