MATH 529: Lecture 7 (02/03/2026)

Today:
- topological invariants
- Euler characteristic (χ)
- genus and ~~cross cap~~ → didn't get to it...

Topological Invariants

We want to define and use efficiently (and easily) computable functions that can help us to distinguish topological spaces of different topological types.

Def A **topological invariant** is a map that assigns the same object to spaces of the same topological type.
↳ usually, a number; but we could also have a "barcode", for instance

Let $f(\cdot)$ be an invariant.

$\mathbb{X} \approx \mathbb{Y} \Rightarrow f(\mathbb{X}) = f(\mathbb{Y})$.

So, $f(\mathbb{X}) \neq f(\mathbb{Y}) \Rightarrow \mathbb{X} \not\approx \mathbb{Y}$ → could be used for contrapositive arguments.

But $f(\mathbb{X}) = f(\mathbb{Y})$ does not necessarily mean $\mathbb{X} \approx \mathbb{Y}$.

If $f(\mathbb{X}) = f(\mathbb{Y}) \Rightarrow \mathbb{X} \approx \mathbb{Y}$, then $f(\cdot)$ is called a **complete invariant**.

Notice that an invariant could assign the same object to spaces of different topological types. The main way to use the invariant is in the contrapositive, i.e., if the invariant is different for a pair of spaces, then the two spaces have different topological types.

While the invariants are defined for topological spaces, we usually use triangulations to compute them. And since they're "invariants", the specific choice of the triangulation is not important.

We introduce a simple to evaluate/compute invariant — the Euler characteristic.

The Euler characteristic (χ) "chi" (originally defined for graphs)

Let K be a simplicial complex, and let s_i be the # i-simplices in K for $0 \le i \le \dim(K)$. Thus,

$$s_i = \left| \{ \sigma \in K \mid \dim \sigma = i \} \right|.$$

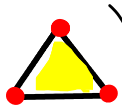
The **Euler characteristic** of K is defined as

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i .$$

Notice that $s_i = 0 \;\; \forall\, i > \dim K$.

Equivalently, $\chi(K) = v - e + f - t + \cdots$, where $v = \#$ vertices, $e = \#$ edges, $f = \#\underbrace{\text{faces}}_{\text{triangles}}$, $t = \#$ tetrahedra, ... and so on.

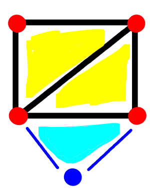
Let us find χ for a triangulation of the closed 2-disc.

χ () $= 3 - 3 + 1 = 1$

\# vertices \# edges \# faces (or \# triangles)

χ is an integer invariant, and it is an **invariant of the underlying space $|K|$**. So, χ is invariant over triangulations of a given space. Thus, any triangulation of a topological space $\mathbb{X}$ has the same $\chi(\mathbb{X})$ value.

Continuing with the example of the disc, we get the same χ using any other triangulation — see two examples below.

χ () $= 4 - 5 + 2 = 1.$

The triangulation is made of two triangles sharing an edge.

Now consider adding one more triangle to get another valid triangulation (as shown in blue).

$\Delta(V) = 1, \Delta(E) = 2, \Delta(F) = 1$, so $\Delta(\chi) = 1 - 2 + 1 = 0!$

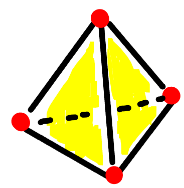
change in \# vertices, edges, triangles

change in χ

Q: Can we use χ to distinguish compact 2-manifolds?

Let us find χ for $\mathbb{S}^2, \mathbb{T}^2, \mathbb{RP}^2$ and $\mathbb{K}^2$, 2-sphere, torus, projective plane, and the Klein bottle.

1. $\mathbb{S}^2$ K:

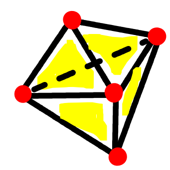


NOT the solid tetrahedron!

Surface of a tetrahedron is a triangulation of $\mathbb{S}^2$.

$\chi(K) = 4-6+4 = 2.$

K':

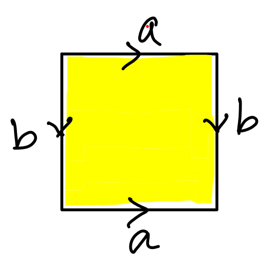


Let us consider another triangulation K' of $\mathbb{S}^2$, made of 3 triangles from top and 3 from bottom joined to form a "sphere".

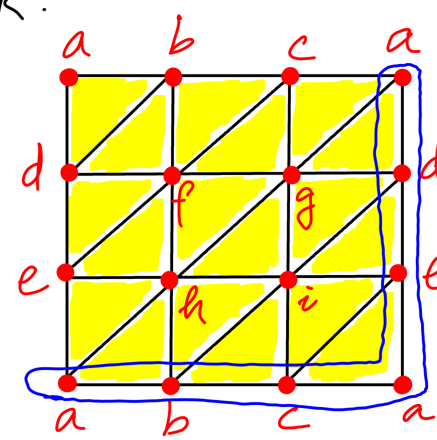
$$\chi(K') = 5-9+6 = 2.$$

Indeed, $\chi(\mathbb{S}^2) = 2$. One could take any triangulation of $\mathbb{S}^2$, χ will be the same.

2. $\mathbb{T}^2$ (torus)



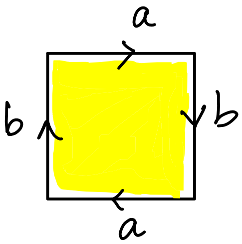
K:

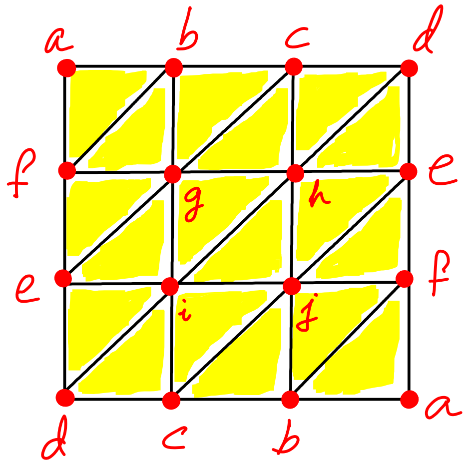


should not double count!

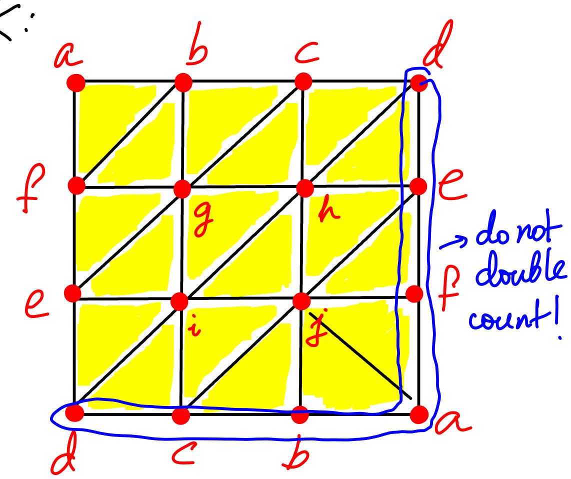
$$\chi(\mathbb{T}^2) = 9-27+18 = 0.$$

3. $\underline{\mathbb{RP}^2}$





K:



$\overline{bf} \in abf, jbf, gbf$! ✗

$\overline{ab} \in abf$! ✗

$\overline{ab} \in abj, abf$ ✓

$\overline{bf} \in bfa, bfg$ ✓

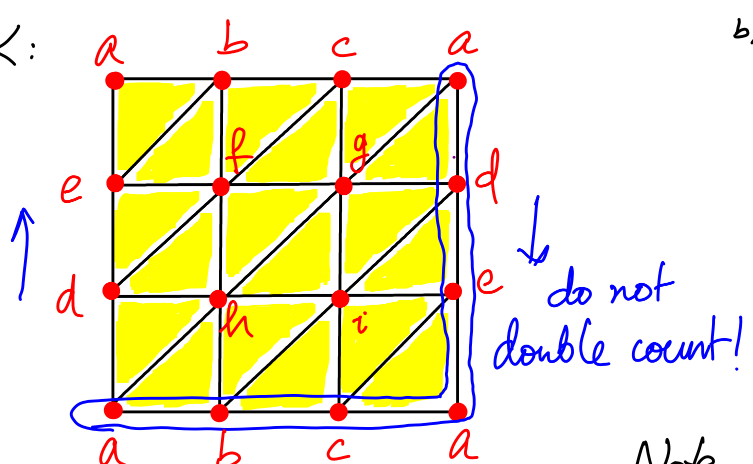
Our first attempt at finding a triangulation (left) of $\mathbb{RP}^2$ is not correct! In particular, edge $\overline{bf}$ (bottom right square) is part of 3 $\triangle$les: abf, bfg, bfj! Also, $\overline{ab}$ and $\overline{af}$ are part of only one triangle each. But $\mathbb{RP}^2$ has no boundary! A correct triangulation is given on the right.

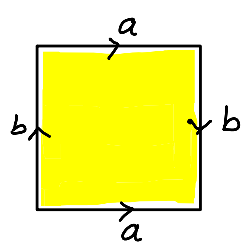
the left triangulation represents $\mathbb{RP}^2$ with a "flap" ($\triangle abf$)

$$\chi(K) = \underbrace{10}_{\text{vertices } a\text{–}j} - 27 + 18 = 1.$$

4. $\mathbb{K}^2$ (Klein bottle)

K:





$$\chi(K) = 9 - 27 + 18 = 0.$$

Note that $\chi(\mathbb{K}^2) = \chi(\mathbb{T}^2)$!

Here is the summary of the χ values we have seen so far.

2-manifold		χ
orientable	S^2	2
	$\mathbb{T}^2$	0
nonorientable	$\mathbb{RP}^2$	1
	$\mathbb{K}^2$	0

So, χ alone is not sufficient to distinguish between all these surfaces!

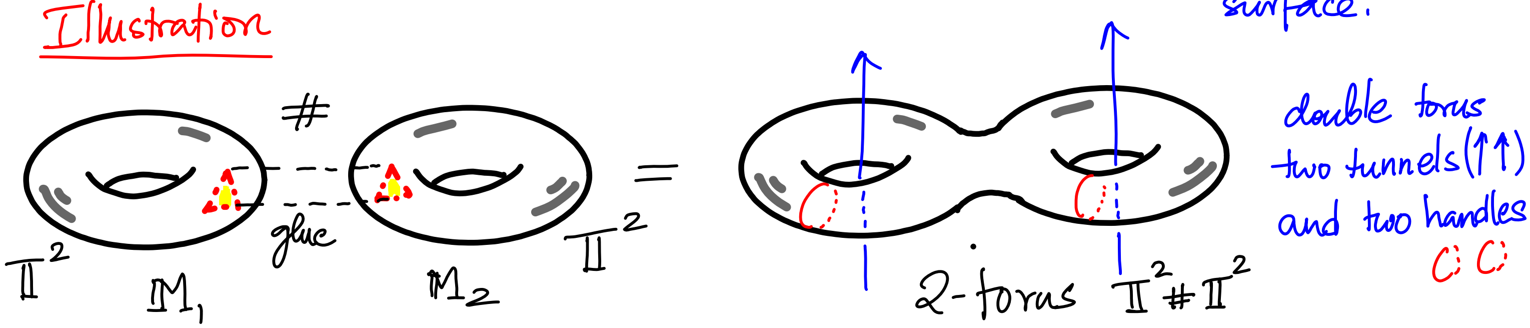
It turns out that if we add orientability to χ, we do get a complete invariant for all compact (connected) 2-manifolds (without boundary). Recall the original classification theorem, which states that every compact connected 2-manifold is homeomorphic to S^2, a connected sum of copies of $\mathbb{T}^2$, or a connected sum of copies of $\mathbb{RP}^2$. With this result in mind, let us first study how χ changes when we take the connected sum of two manifolds.

Theorem For compact, connected surfaces (2-manifolds) $\mathbb{M}_1$ and $\mathbb{M}_2$,

$$\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2.$$

take one Δle out of each surface.

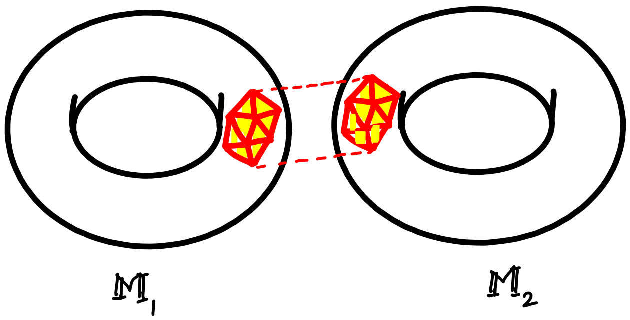
Illustration



We remove a triangle each from both $\mathbb{M}_1$ and $\mathbb{M}_2$, and glue along the boundaries of these triangles.

$\Delta(V) = -3$, $\Delta(E) = -3$, $\Delta(F) = -2$. So, $\Delta(\chi) = -3 - (-3) + (-2) = -2$.

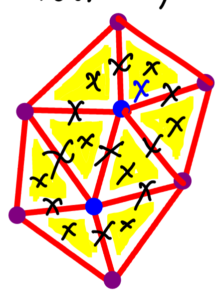
The result holds for the removal of a disc in general, and not just for the case of (the removal of) a triangle.



Here, we remove a patch (homeomorphic to the disc) from both $\mathbb{M}_1$ and $\mathbb{M}_2$, and identify the boundaries, which is composed of 6 edges and 6 vertices (to form a loop).

From the middle regions of each patch, we remove 2 vertices, 9 edges, and 8 triangles.

The change in $\chi(\mathbb{M}_1 \# \mathbb{M}_2)$ contributed by the simplices removed from $\mathbb{M}_1$ is



simplices marked with an 'x' are removed, and so are the two middle vertices.

$-(2-9+8) = -1$. A same change is contributed by the simplices removed from $\mathbb{M}_2$.

There is no change in χ from identifying the boundaries — as they are both cycles (so have same # of vertices & edges).

As such, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

We get the same result even if we were to remove different "discs" from the two tori. Just that the homeomorphism defining the gluing would be more complicated there.

We could prove this result in general (for a removal of a general disk).

Theorem Two compact, closed, connected 2-manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$ are homeomorphic **if and only if**

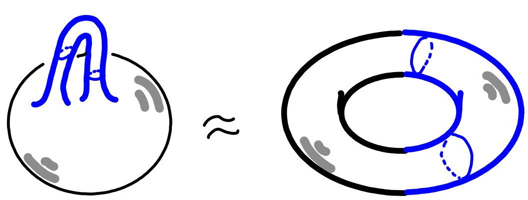
1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ and
2. either $\mathbb{M}_1$ and $\mathbb{M}_2$ are both orientable, or are both nonorientable.

We could perform both checks 1 and 2 efficiently on a computer.

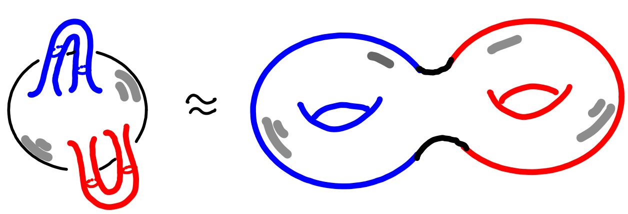
in polynomial time, to be precise

Genus and Cross-cap → Two more terms used in the context of 2-manifolds.

Def The connected sum of g tori is called a surface with **genus** g. Equivalently, a 2-sphere with 1 tube is a surface with genus $g=1$.

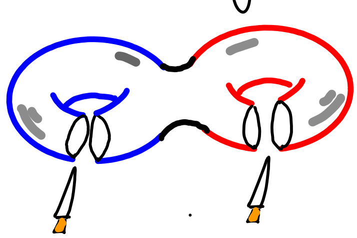


Sphere with one tube is homeomorphic to torus. (so, torus has genus = 1).



A sphere with two tubes is homeomorphic to the double torus.

$\mathbb{M}$ has genus g ⟹ there are g disjoint closed curves on $\mathbb{M}$ along which you can cut without disconnecting $\mathbb{M}$.



$g=2$ here. If we cut along one more closed curve now, we get two pieces that are disconnected.