MATH 529 : Lecture 8 (02/05/2026)

Today:
* cross caps
* orientation of a simplex
* orienting surfaces.

Recall The connected sum of g tori is an orientable surface with genus g.

How is the Euler characteristic connected to genus?

Euler characteristic and Genus

Recall $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

$$\chi(g\mathbb{T}^2) = 2 - 2g.$$

connected sum of g tori

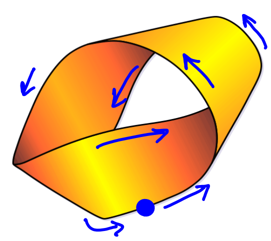
We could easily prove this result using induction, using the above fact about $\chi(\mathbb{M}_1 \# \mathbb{M}_2)$, and $\chi(\mathbb{T}^2) = 0$.

Recall, $\chi(\mathbb{S}^2) = 2$ Also, $\chi(\#(\mathbb{T}^2)^g)$

Cross cap

Recall that the Möbius strip has only one edge, i.e., its boundary is a single circle.

Starting from a point on the edge, we can traverse the entire boundary to come back to the same point (as shown by arrows).



(image: www)

If we remove an open disc from the 2-sphere, and glue a Möbius strip along its edge onto the boundary of this disc, we have added one **cross cap**.

A sphere with a single cross cap is homeomorphic to the real projective plane ($\mathbb{RP}^2$)
A sphere with two cross caps is homeomorphic to the Klein bottle ($\mathbb{K}^2$).
In general, a sphere with g cross caps is the connected sum of g projective planes, and we have

$$\chi(g\mathbb{RP}^2) = 2-g.$$

also, sphere with g cross caps

On the other hand, the classification theorem for compact connected 2-manifolds says that any non-orientable surface (2-manifold) is homeomorphic to the connected sum of copies of $\mathbb{RP}^2$, the projective plane. Recall that once we glue at least one cross cap, the surface becomes non-orientable. We could use the above result relating χ and the # cross caps and the result on how χ changes when we take the connected sum of two surfaces to identify the # copies of $\mathbb{RP}^2$ whose connected sum is homeomorphic to a given nonorientable surface.

Consider $\mathbb{T}^2 \# \mathbb{RP}^2$. We get

$$\chi(\mathbb{T}^2 \# \mathbb{RP}^2) = \chi(\mathbb{T}^2) + \chi(\mathbb{RP}^2) - 2 = 0+1-2 = -1.$$

To get the # cross caps in the homeomorphic surface, we set

$$\chi(g\mathbb{RP}^2) = 2-g = -1 \implies g=3.$$

Hence we should have

$$\mathbb{T}^2 \# \mathbb{RP}^2 \approx \#(\mathbb{RP}^2)^3,$$

as we stated in Lecture 5!

Moving on, we now consider orientations of simplices, and how to extend them to possibly orient entire simplicial complexes.

Def Let σ be a simplex (geometric or abstract). Two orderings of its vertices are equivalent if they differ by an even permutation. If $\dim(\sigma) > 0$, the orderings fall into **two** equivalence classes. Each equivalence class is an **orientation** of σ.

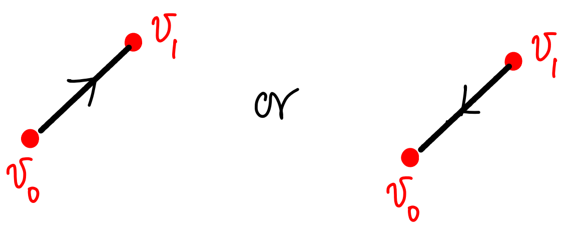
A 0-simplex has only one orientation.

An **oriented simplex** is a simplex σ with an orientation of σ.

An even permutation is obtained by doing an even number of pairwise swaps.

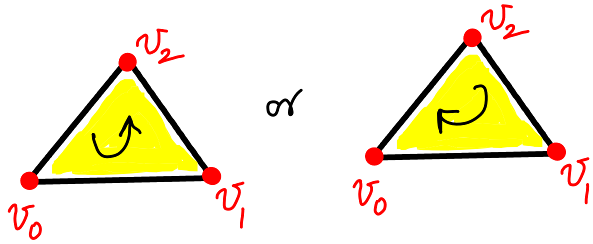
Notation For σ with vertices $\{v_0, \ldots, v_k\}$, $\sigma = [v_0, \ldots, v_k]$ denotes an oriented simplex. Note that we use σ to denote both the default and the oriented simplex.

Examples



1-simplex

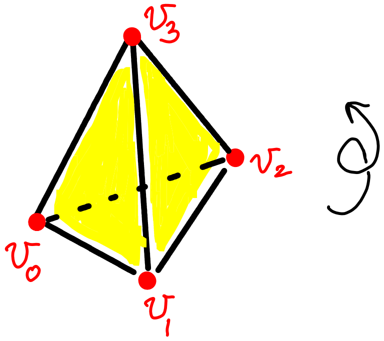
$[v_0 v_1]$ is opposite to $[v_1 v_0]$ (we can go forward or backward)



$[v_0 v_1 v_2]$ is the same orientation as $[v_2 v_0 v_1]$ — both go CCW (↺). But $[v_1 v_0 v_2]$ goes clockwise.

$[v_0 v_1 v_2]$ and $[v_2 v_0 v_1]$ are same orientations

$[v_0 v_1 v_2] \longrightarrow [v_0 v_2 v_1] \longrightarrow [v_2 v_0 v_1]$ two swaps, so they are even permutations of each other.



$[v_0 v_1 v_2 v_3]$ is the same orientation as $[v_2 v_0 v_1 v_3]$ while $[v_0 v_2 v_1 v_3]$ is the opposite orientation.

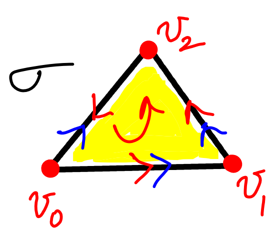
"convex hull" here is used in general to cover both geometric and abstract cases.

Induced orientation

Let σ have vertices $\{v_0, \dots, v_k\}$. When σ is oriented, it induces an "induced orientation" on all its $(k-1)$-faces. Each $(k-1)$-face of σ can be denoted as conv$\{v_0, \dots, \hat{v}_i, \dots, v_k\}$, where the '^' (hat) above a vertex indicates that it is excluded.

Let σ be oriented as $[v_0, \dots, v_k]$. Then the orientation induced by σ on $\tau = \text{conv}\{v_0, \dots, \hat{v}_i, \dots v_k\}$ is the same as that of $[v_0, \dots, \hat{v}_i, \dots, v_k]$ if i is even. Else, it is the opposite orientation.

For illustration, consider the triangle oriented as $\sigma = [v_0 v_1 v_2]$.



σ has three edges as faces, given by $\{v_0, v_1\}$, $\{v_0, v_2\}$, and $\{v_1, v_2\}$.

The induced orientations on these edges are $[v_0 v_1]$, $[v_2 v_0]$, and $[v_1 v_2]$, respectively.

"$>$": induced orientations

"$>$": edges' own orientations, which are independent of the induced orientations

We leave out v_1 to get the edge $\{v_0, v_2\}$, and hence its induced orientation is the reverse of $[v_0 v_2]$, i.e., it is $[v_2 v_0]$.

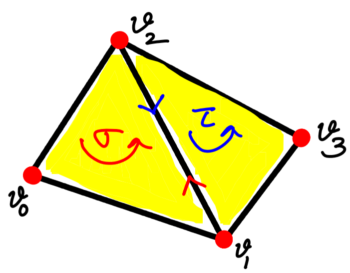
In the case of a triangle, the induced orientations indeed agree with the intuition of arrows "induced" on the edges by the CCW (or CW) arrow of the triangle.

counterclockwise

Comparing Orientations

Let σ, τ be simplices. If $\dim \sigma \neq \dim \tau$ we cannot compare their orientations. So let us consider the case when $\dim \sigma = \dim \tau = k$.

If σ and τ share a common $(k-1)$-face, they are **consistently oriented**, or oriented the same way, if they induce **opposite** orientations on the common $(k-1)$-face.



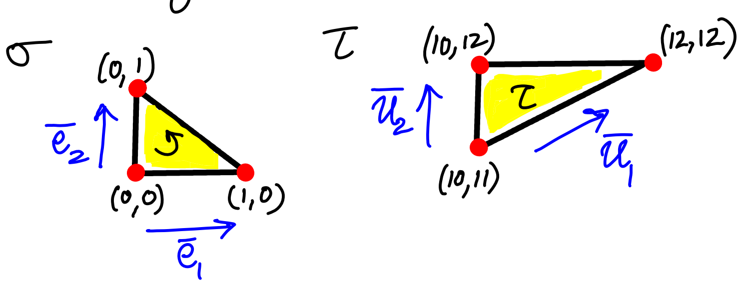
(induced orientations on $\overline{v_1 v_2}$ shown by $\wedge$ and $\vee$)

orientations induced by $\sigma = [v_0 v_1 v_2]$ and $\tau = [v_1 v_3 v_2]$ on $\overline{v_1 v_2}$ are opposite.

Hence the two triangles are consistently oriented — both CCW here.

Note that the induced orientations are separate from the edge's own orientation. Here, we could have $[v_1 v_2]$ or $[v_2 v_1]$ as the inherent orientation of $\overline{v_1 v_2}$. The induced orientations are still as shown in the figure.

Note: We could compare orientations of two k-simplices even if they do not share a common $(k-1)$-face, if they both are sitting in the same k-dimensional plane. For instance, consider two disjoint triangles in $\mathbb{R}^2$.



The 3 vertices of a triangle generate two vectors, whose cross-product can be used to calculate the **signed area** of the triangle.

For oriented triangle $[v_0 v_1 v_2]$, consider vectors $\bar{e}_1 = \bar{v}_1 - \bar{v}_0$ and $\bar{e}_2 = \bar{v}_2 - \bar{v}_0$.

For σ, we can take $\bar{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\bar{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

The signed area of σ is given by $\text{area}(\sigma) = \frac{1}{2}|\underbrace{\bar{e}_1 \times \bar{e}_2}_{\text{area of the parallelogram generated by } \bar{e}_1 \,\&\, \bar{e}_2}| = \frac{1}{2}\det([\bar{e}_1 \bar{e}_2])$.

Thus, $\text{area}(\sigma) = \frac{1}{2}\left|\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right| = \frac{1}{2}$.

determinant

Similarly for τ, we choose $\bar{u}_1 = \begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}10\\11\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\bar{u}_2 = \begin{bmatrix}10\\12\end{bmatrix} - \begin{bmatrix}10\\11\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$,

giving $\text{area}(\tau) = \frac{1}{2}\left|[\bar{u}_1, \bar{u}_2]\right| = \frac{1}{2}\left|\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}\right| = 1$.

Since σ and τ have the same sign for their signed areas, they are consistently oriented. In this case, they are both oriented CCW. In fact, both sets of vectors $\{\bar{e}_1, \bar{e}_2\}$ and $\{\bar{u}_1, \bar{u}_2\}$ orient $\mathbb{R}^2$ in the same way here.

These computations are naturally extended to d-dimensions for $d \geq 3$. We can compute the signed d-volume in the same fashion.

We could compare orientations in the abstract setting as well.

Consistently oriented simplices

We consider an example in the abstract setting.

Let $\sigma_1 = [2\ 5\ 12\ 19]$ and $\sigma_2 = [12\ 19\ 7\ 2]$ be two oriented 3-simplices. Are they consistently oriented?

Notice that $\tau = \{2, 12, 19\}$ is the common 2-face.

exclude to get τ

$$\sigma_1 = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 2 & 5 & 12 & 19 \end{array} \qquad \sigma_2 = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 12 & 19 & 7 & 2 \end{array}$$

We get τ by removing the first $(i=1)$ vertex from σ_1. Similarly, we get τ by removing the second $(i=2)$ vertex from σ_2.

Hence, the orientation induced on τ by σ_1 is $[12\ 2\ 19]$, which is the opposite orientation to $[2\ 12\ 19]$. differ by 1 pairwise swap

The orientation induced on τ by σ_2 is $[12\ 19\ 2]$.

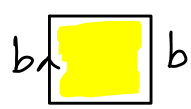
The two induced orientations are opposite here. Hence σ_1 and σ_2 are consistently oriented.

We extend the idea of when two d-simplices are consistently oriented to when the entire triangulation is consistently oriented.

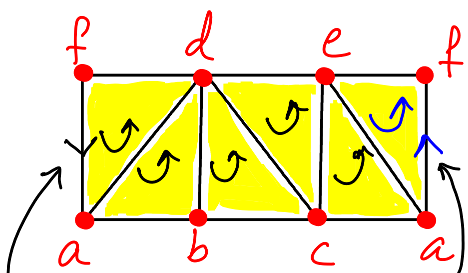
Def A triangulable d-manifold (with or without boundary) is called **orientable** if all the d-simplices in any triangulation of the manifold can be consistently oriented. Else, it is a **nonorientable** manifold.

Examples

1. Cylinder



diagram



triangulation

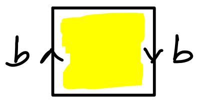
all triangles oriented CCW.

induced orientations are opposite

It can be checked that the orientations induced by each pair of triangles on their common (shared) edge are indeed opposite. In particular, notice that this is indeed the case for $\overline{af}$ — induced orientations from $[adf]$ and $[afe]$ are $[fa]$ and $[af]$, respectively.

Thus, the cylinder is orientable.

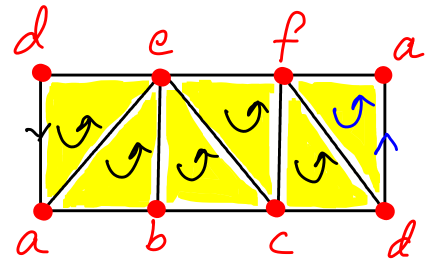
2. Möbius strip



diagram

We had noted previously that the Möbius strip is non-orientable.

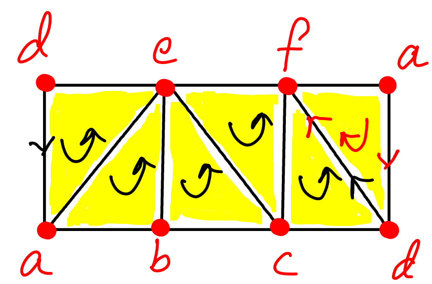
triangulation →



The orientations induced on edge $\overline{ad}$ by [aed] and [afd] are the same here — both being [da], or $d \to a$. Thus, Δaeb and Δadb are not consistently oriented.

Notice that the induced orientations on all remaining shared edges except $\overline{ad}$ are indeed opposite — check the induced orientations on $\overline{ae}$, $\overline{be}$, $\overline{ce}$, $\overline{cf}$ and $\overline{df}$.

If we fix the orientations such that induced orientations on $\overline{ad}$ are opposite, say, by orienting Δadf clockwise, i.e., [adf], then the induced orientations on $\overline{df}$ are now identical — [df].



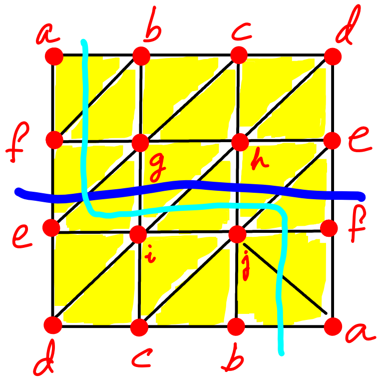
It turns out that however we orient the triangles, the induced orientations will be the same on one of the shared edges. As such, all triangles in the Möbius strip cannot be consistently oriented.

Hence the Möbius strip is non-orientable.

In fact, Möbius strips are the minimal non-orientable "~~objects~~" in 2D. For instance, you can identify Möbius strips in the ~~triangulations~~ of $\mathbb{RP}^2$ and $\mathbb{K}^2$ we introduced in Lecture 7!

Hence for surfaces, it is sufficient to identify a Möbius strip in the given triangulation to "certify" its nonorientability.

$\mathbb{RP}^2$:



The middle strip of 6 triangles forms a Möbius strip! And there are several other instances of the Möbius strip here — another instance is identified by ~ curve.

In both the cylinder and the Möbius strip, the boundary edges can be oriented arbitrarily. These are the edges that are faces of only one triangle each. For example, in the Möbius strip above, edges $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{de}$, $\overline{ef}$ and $\overline{af}$ are boundary edges, and they can be assigned orientations arbitrarily without affecting the (non-)orientability of the manifold.