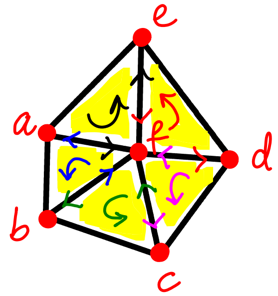
MATH 529: Lecture 9 (02/10/2026)

Today:
- ∗ propagating orientation
- ∗ subdivision
- ∗ star, closed star, link

To algorithmically check the orientability of a manifold, we try to propagate an orientation.

Checking orientability of a d-manifold Start by assigning an orientation to one d-simplex σ (pick one of the two possibilities). Then "propagate" this orientation to any other d-simplex σ' that shares a common (d-1)-simplex τ, say, with σ. In other words, orient σ' such that the orientations induced on τ by σ and σ' are opposite. Continue this process until all d-simplices are oriented. If we can consistently orient all d-simplices, the manifold is orientable. Else, it is non-orientable.

Example:



Start with [afe], and propagate this orientation in the order [abf], [bcf], [cdf], and [def]. Notice that the shared edges $\overline{af}$, $\overline{bf}$, $\overline{cf}$, $\overline{df}$, and $\overline{ef}$ have opposite induced orientations.

To certify non-orientability of a 2-manifold, it's best to identify a Möbius strip in it (as a subcomplex).

We now define several constructs on simplicial complexes that are simplicial analogues of standard constructs on (continuous) spaces, such as subspaces, open sets, neighborhoods of points, boundary of a neighborhood, etc.

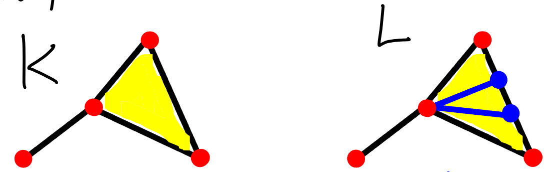
Subdivision

Def A simplicial complex L is a **subdivision** of another complex K if $|K| = |L|$, and $\forall \sigma \in L$, $\sigma \subseteq \tau \in K$.

not $\approx$!

In words, every simplex in L is contained in a simplex in K.

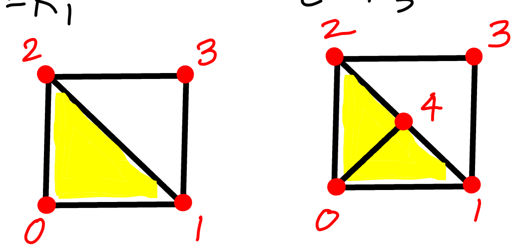
Example:



the 3 triangles are contained in the one triangle in K, for instance.

Here is another example we saw in a different context (in Lecture 6):

$K = K_1$ $\qquad$ $L = K_3$



Barycentric Subdivision

One way to create a subdivision of K is by forming the **barycentric subdivision**, denoted $\mathrm{Sd}\,K$ (this is a "standard" way to subdivide a complex).

The **barycenter** of a simplex is the centroid of its vertices.

We define (or construct) the barycentric subdivision inductively on the dimension of the simplices.

Inductive construction of Sd K

all simplices in K with dimension $\leq j$

Notation $K^{(j)} = \{\sigma \in K \mid \dim \leq j\}$ is the j-skeleton of K.

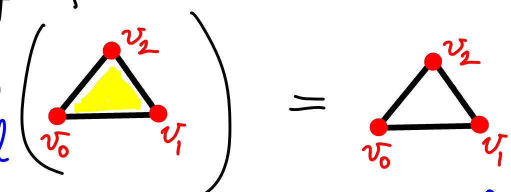
Thus, $K^{(0)}$ is the set of vertices of K.

We start by defining $\mathrm{Sd}\,K^{(0)} = K^{(0)}$. (barycenter of a vertex is the vertex itself).

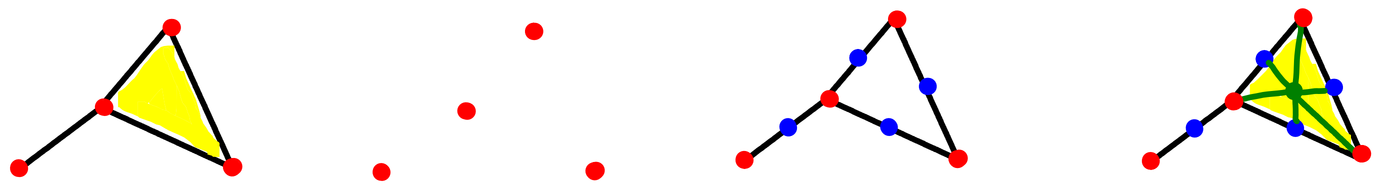
If we have $\mathrm{Sd}\,K^{(j-1)}$, we construct $\mathrm{Sd}\,K^{(j)}$ by adding the barycenter of every j-simplex as a new vertex, and connecting that vertex to each simplex that subdivides the boundary of that simplex.

$$\mathrm{Bd}\,\sigma = \bigcup_{\tau < \sigma} \tau$$

∂ or Bd

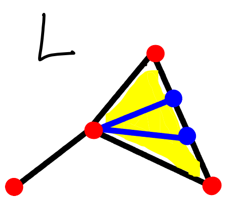


We illustrate this construction on the same simplicial complex we used to illustrate subdivisions.



K $\qquad$ $\mathrm{Sd}\,K^{(0)} = K^{(0)}$ $\qquad$ $\mathrm{Sd}\,K^{(1)}$ $\qquad$ $\mathrm{Sd}\,K^{(2)} = \mathrm{Sd}\,K$

Note that $Sd\, K^{(0)} = K^{(0)}$ as the barycenter of a vertex is the vertex itself. Each edge in K is replaced by 2 edges, while each triangle is replaced by 6 triangles in $Sd\, K$.



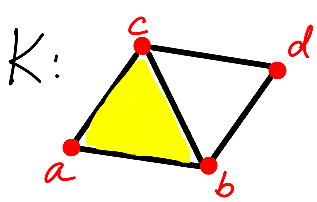
In the previous example, L is a subdivision of of K, but is not a barycentric subdivision.

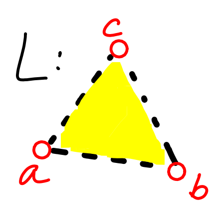
In certain applications, we may want not to subdivide certain simplices, e.g., keep a subset of original edges in tact.

We now present definitions on simplicial complexes corresponding to open neighborhoods around points in $\mathbb{R}^d$. We start with some preliminary definitions.

Def A **subcomplex** of K is a simplicial complex L, such that $L \subseteq K$.
A subset that is a simplicial complex by itself

Def The smallest subcomplex containing a subset $L \subseteq K$ is its **closure**, $Cl\, L = \{\tau \in K \mid \tau \le \sigma \in L\}$. $\to$ could also use $\overline{L}$





L is not a subcomplex of K here!

$Cl\, \{\overline{ac}, d\} = \{\overline{ac}, d, a, c\}$. $\quad Cl\, \{\Delta abc\} = \{\Delta abc, \overline{ab}, \overline{ac}, \overline{bc}, a, b, c\}$.

Notice that $Cl\, L$ is a simplicial complex by itself.

Def For a simplex $\sigma \in K$, we define its **boundary** and **interior** as follows.

$$Bd\, \sigma = \bigcup_{\tau < \sigma} \tau \quad \text{and} \quad Int\, \sigma = \sigma \setminus Bd\, \sigma.$$

or $\partial\sigma$ $\qquad$ or $\mathring{\sigma}$

Def For a vertex $\bar{v}$ of K, the **star** of $\bar{v}$, denoted $St\,\bar{v}$, is

$$St\,\bar{v} = \bigcup_{\sigma \geq \bar{v}} \operatorname{Int} \sigma.$$

coface

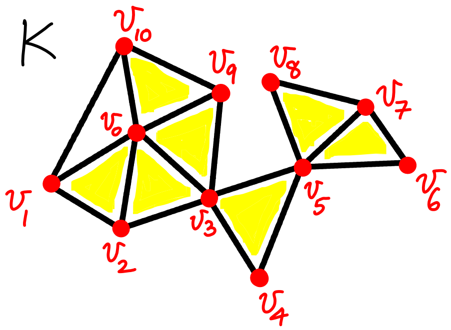
Union of interiors of all simplices that have $\bar{v}$ as a vertex

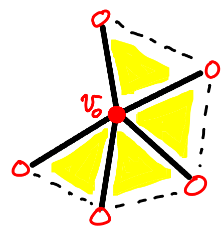
Also, $Cl\,St\,\bar{v}$ (or $\overline{St\,\bar{v}}$) is the **closed star** of $\bar{v}$

take the closure of $St\,\bar{v}$, i.e., throw in all faces.

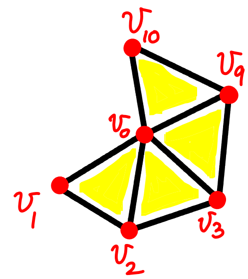
The set $Cl\,St\,\bar{v} \setminus St\,\bar{v}$ is the **link** of $\bar{v}$, denoted $Lk\,\bar{v}$.

Here is an example — consider the complex K shown.

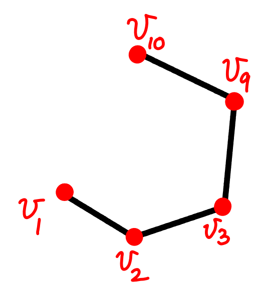




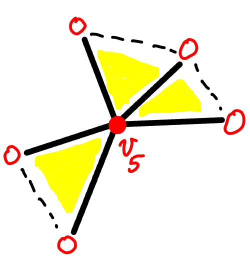
$St\,v_0$



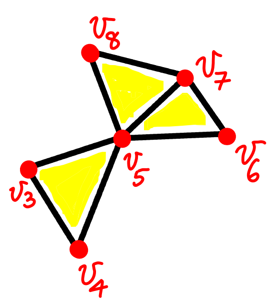
$Cl\,St\,v_0$



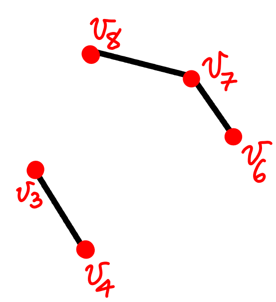
$Lk\,v_0$



$St\,v_5$



$Cl\,St\,v_5$



$Lk\,v_5$

Following these examples, we could make the following observations.

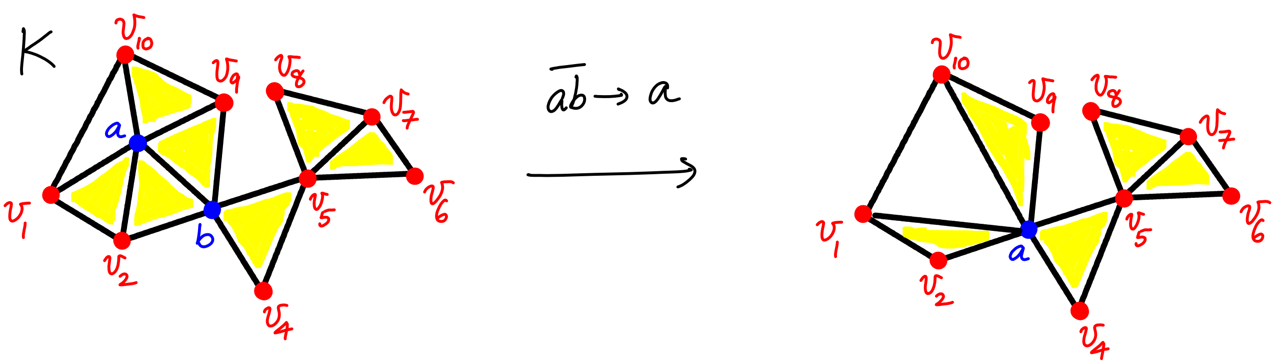
$St\,\bar{v}$ and $Cl\,St\,\bar{v}$ are both path connected, but $Lk\,\bar{v}$ need not be be connected.

there is a path between any two points in the set

Also, $Cl\,St\,\bar{v}$ and $Lk\,\bar{v}$ are subcomplexes of K, but $St\,\bar{v}$ is typically not a subcomplex.

An aside on mesh simplification

Links of vertices are employed in mesh simplification — you want a simplicial complex with a smaller number of simplices while preserving topology. A standard operation used in this context is edge contraction, where we replace edge $\overline{ab}$, say, with vertex a. We also make associated changes to other simplices connected to edge $\overline{ab}$. Here is an illustration.



Here, contracting $\overline{ab}$ to a preserves the topology — at least, we do not close any holes in K. But if we were to contract edge $\overline{v_1 v_{10}}$, we will close a hole!

We could define a condition on how the links of a, b, and of $\overline{ab}$ (to be defined next) are related. This is a local condition, and can be checked quickly. If it is satisfied, we can contract $\overline{ab}$ without the fear of closing any holes.

Such operations are critical to efficient computations on simplicial complexes. We will talk about them later in the semester.

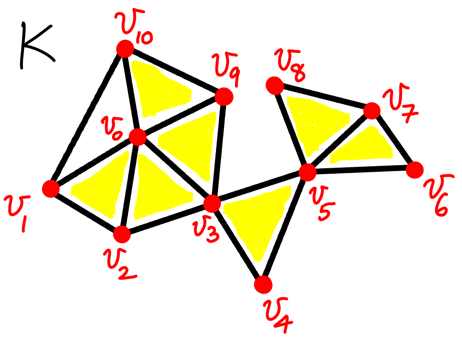
We now extend the definitions of star and link to simplices (of any dimension) and then to collections of simplices.

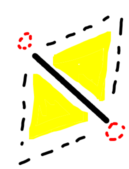
Def For a simplex $\sigma \in K$, we let

$$\operatorname{St}\sigma = \bigcup_{\tau \geq \sigma} \operatorname{Int}\tau, \qquad \operatorname{Cl}\operatorname{St}\sigma \text{ is its closed star,}$$

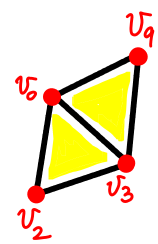
$$\text{and} \quad \operatorname{Lk}\sigma = \left\{ \bigcup \tau \mid \tau \in \operatorname{Cl}\operatorname{St}\sigma,\ \tau \cap \sigma = \emptyset \right\}.$$

In words, the link of σ is the set of simplices in its closed star which are disjoint from σ.

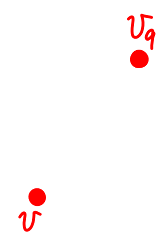




$\operatorname{St}\overline{v_0v_3}$



$\operatorname{Cl}\operatorname{St}\overline{v_0v_3}$



$\operatorname{Lk}\overline{v_0v_3}$