MATH 565: Lecture 4 (01/22/2026)

Today:
- illustration of gradient descent
- local optimality in d-dimensions
- convex functions

We saw gradient descent for $f(x) = x^2 \sin(x) + x$, with
$f'(x) = 2x\sin x + x^2\cos(x) + 1$,
and $f''(x) = (2-x^2)\sin(x) + 4x\cos(x)$

A 2D Extension

Consider $G(x,y) = f(x) + f(y)$, which is a separable function, i.e., it has no terms involving both x and y.

$$\Rightarrow \nabla G = \begin{bmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x\sin(x) + x^2\cos(x) + 1 \\ 2y\sin(y) + y^2\cos(y) + 1 \end{bmatrix}$$

Gradient descent now implements $\bar{w} \leftarrow \bar{w} - \alpha \nabla G(\bar{w})$.
Check Python notebook on the course web page...

We consider one final example with a nonseparable function:
Let $d(x,y) = x^2\sin(y)$, which gives

$$\nabla d = \begin{bmatrix} 2x\sin y \\ x^2\cos y \end{bmatrix}.$$

The point of these exercises was to demonstrate that optimization using gradient descent in high dimensions could become quite complex quickly...

More on local optimality in d dimensions

The sufficient conditions (first and second order) for local optimality in d dimensions (given in Lemma 2 in Lecture 3) can be qualified further to specify more nuanced cases for the second order condition.

Recall: $\nabla J = \bar{0}$ (first order optimality condition)

$HJ \succ 0$, i.e., H is PD (positive definite) (second order optimality condition)

(Result: a real symmetric matrix H is PD $\Leftrightarrow$ all its eigenvalues are > 0).

Second order optimality conditions

Let $\nabla J(\bar{w}_0) = \bar{0}$.

1. If $HJ(\bar{w}_0) \succ 0$, then $\bar{w}_0$ is a local minimum (Lemma 2)
   (H is PD)
2. If $HJ(\bar{w}_0) \prec 0$, then $\bar{w}_0$ is a local maximum.
   (H is ND, negative definite)
3. If $HJ(\bar{w}_0)$ is indefinite, then $\bar{w}_0$ is a saddle point.
4. If $HJ(\bar{w}_0) \succeq 0$ or $HJ(\bar{w}_0) \preceq 0$, then the test is inconclusive. positive/negative semidefinite

The presence of local optima can make the task of finding a global optimum hard. We now study a class of functions for which any local optima are also guaranteed to be global optima — convex functions.

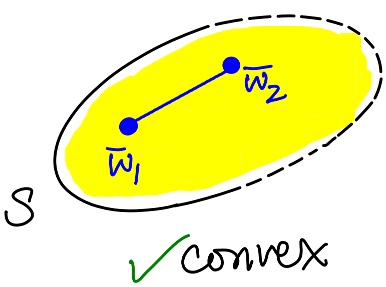
Convex Sets and Functions

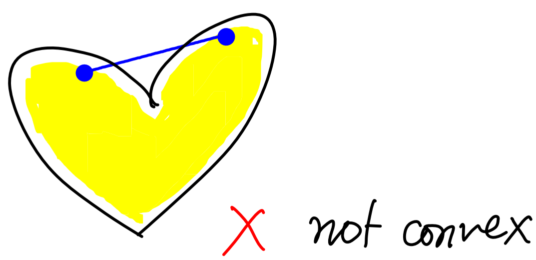
We first define convex sets.

Def A set $S \subseteq \mathbb{R}^d$ is **convex** if $\forall\, \bar{w}_1, \bar{w}_2 \in S$,

$$\lambda \bar{w}_1 + (1-\lambda)\bar{w}_2 \in S \quad \forall\, \lambda \in [0,1].$$

In words, the line segment connecting $\bar{w}_1$ and $\bar{w}_2$ lies in S.



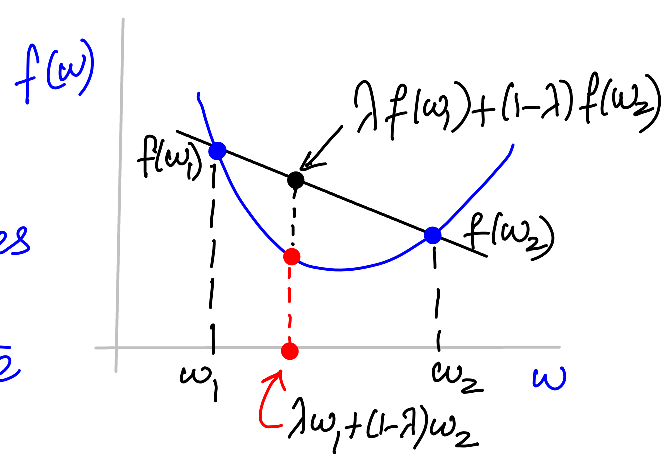


Def A function $f: \Omega \to \mathbb{R}$ is a **convex function** for convex set $\Omega \subseteq \mathbb{R}^d$ if

$$f(\lambda \bar{w}_1 + (1-\lambda)\bar{w}_2) \leq \lambda f(\bar{w}_1) + (1-\lambda) f(\bar{w}_2) \qquad (*)$$

$$\forall\, \bar{w}_1, \bar{w}_2 \in \Omega, \ \forall\, \lambda \in [0,1].$$

In 1D, the value of the function at a convex combination of two points lies not above the corresponding convex combination of the function values at the two points.



Proposition 3 If $f:\Omega\to\mathbb{R}$ is convex, then

$$f\left(\sum_{i=1}^{k}\lambda_i\bar{\omega}_i\right)\leq\sum_{i=1}^{k}\lambda_i f(\omega_i) \text{ for } \omega_i\in\Omega,\ 0\leq\lambda_i\leq 1,\ \sum_{i=1}^{k}\lambda_i=1.$$

Proof We use induction on k.

$k=1$ is trivial: $\lambda_1=1$ and $f(1\bar{\omega}_1)=1f(\bar{\omega}_1)$

base case $k=2$ is $(*)$ from the definition of a convex function.

Assume result holds for k, and consider statement for $k+1$:

We want to show that

$$f\left(\sum_{i=1}^{k+1}\lambda_i\bar{\omega}_i\right)\leq\sum_{i=1}^{k+1}\lambda_i f(\bar{\omega}_i) \quad\text{holds.}$$

Assume $0<\lambda_{k+1}<1$ $(\lambda_{k+1}=0 \text{ or } 1\Rightarrow \text{trivial})$

note: $\sum_{i=1}^{k}\lambda_i=1-\lambda_{k+1}$

$$\Rightarrow f\left(\sum_{i=1}^{k}\lambda_i\bar{\omega}_i+\lambda_{k+1}\bar{\omega}_{k+1}\right)=f\left(\frac{1-\lambda_{k+1}}{(1-\lambda_{k+1})}\sum_{i=1}^{k}\lambda_i\bar{\omega}_i+\lambda_{k+1}\bar{\omega}_{k+1}\right)$$

$$=f\left((1-\lambda_{k+1})\left[\sum_{i=1}^{k}\lambda_i'\bar{\omega}_i\right]+\lambda_{k+1}\bar{\omega}_{k+1}\right) \quad\text{for } \lambda_i'=\frac{\lambda_i}{1-\lambda_{k+1}} \to 0\leq\lambda_i'\leq 1$$

$$\leq(1-\lambda_{k+1})f\left(\sum_{i=1}^{k}\lambda_i'\bar{\omega}_i\right)+\lambda_{k+1}f(\bar{\omega}_{k+1}) \quad\text{by result for } k=2 \text{ (base case)}$$

But $\sum_{i=1}^{k}\lambda_i'=1$, which gives that the expression is

$$\leq(1-\lambda_{k+1})\sum_{i=1}^{k}\lambda_i'f(\bar{\omega}_i)+\lambda_{k+1}f(\bar{\omega}_{k+1}) \quad\text{by induction assumption}$$

$$=\sum_{i=1}^{k+1}\lambda_i f(\bar{\omega}_i) \qquad \text{as } (1-\lambda_{k+1})\lambda_i'=\lambda_i$$

$\square$

Useful Properties of Convex Functions

1. If f_1, f_2 are convex functions, then $g = f_1 + f_2$ is convex.
   (sum of convex functions is convex)

2. If f_1, f_2 are convex functions, then $g = \max(f_1, f_2)$ is convex.
   (max of convex functions is convex)

3. If $f : \Omega \to \mathbb{R}$ is convex and $f(\bar{w}) \geq 0$, then $g = f^2 = [f(\bar{w})]^2$ is convex.

4. If $f : \mathbb{R} \to \mathbb{R}$ is convex and $g : \mathbb{R}^d \to \mathbb{R}$ is linear, then $h : \mathbb{R}^d \to \mathbb{R}$ defined as $h = f(g(\bar{w}))$ is convex.

Proof

$$f(\lambda w_1 + (1-\lambda) w_2) \leq \lambda f(w_1) + (1-\lambda) f(w_2) \qquad (1)$$

Let $g(\bar{w}) = \bar{w}^T \bar{x} + b$ linear function.

Then, $h(\bar{w}) = f(g(\bar{w})) = f(\bar{w}^T \bar{x} + b)$.

$$\Rightarrow h(\lambda \bar{w}_1 + (1-\lambda) \bar{w}_2) = f(g(\lambda \bar{w}_1 + (1-\lambda) \bar{w}_2))$$

$$= f(\bar{x}^T (\lambda \bar{w}_1 + (1-\lambda) \bar{w}_2) + \underbrace{b}_{(\lambda + (1-\lambda)) b})$$

$$= f(\lambda \underbrace{(\bar{x}^T \bar{w}_1 + b)}_{w_1} + (1-\lambda) \underbrace{(\bar{x}^T \bar{w}_2 + b)}_{w_2})$$

$$\leq \lambda f(\bar{x}^T \bar{w}_1 + b) + (1-\lambda) f(\bar{x}^T \bar{w}_2 + b) \quad \text{by (1)}$$

$$= \lambda h(\bar{w}_1) + (1-\lambda) h(\bar{w}_2)$$

$\square$

This form of composition of functions is used in many machine learning contexts, e.g., in deep learning networks, a node may take the input from k other nodes, combine them in a linear or affine format, and then apply a convex transformation to that combination.

The order in which the composition is taken in the above statement is important! For instance, if we consider $f(x)=x^2$, which is convex, and $g(x)=-x$, which is linear, and then consider

$$j(x)=g(f(x))=-x^2,$$

which is concave!