MATH 565: Lecture 5 (01/27/2026)

Today: * More results on convexity
* Details of gradient descent

Recall: $f: \Omega \to \mathbb{R}$ for convex $\Omega \subseteq \mathbb{R}^d$ is convex if $\forall \bar{w}_1, \bar{w}_2 \in \Omega$, $\lambda \in [0,1]$,

$$f(\lambda \bar{w}_1 + (1-\lambda)\bar{w}_2) \leq \lambda f(\bar{w}_1) + (1-\lambda) f(\bar{w}_2) \quad \text{——— } (*)$$

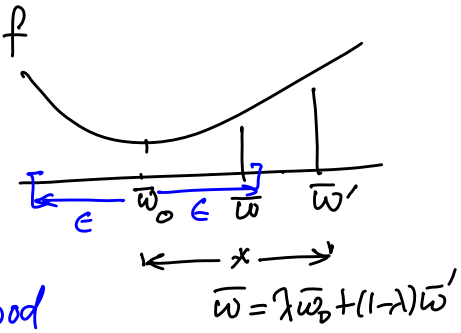
We can use convexity to show that every local minimum of a convex function must be a global minimum!

Proof

Let $\bar{w}_0$ be a local minimum of $f(\bar{w})$.

$$\Rightarrow f(\bar{w}) \geq f(\bar{w}_0) \quad \forall \|\bar{w} - \bar{w}_0\| \leq \epsilon \quad \text{——— (1)}$$

↘ small enough neighborhood of $\bar{w}_0$



Consider $\bar{w}'$ such that $\|\bar{w}' - \bar{w}_0\| > \epsilon$.

Let $\lambda \in (0,1)$ be such that

$$\bar{w} = \lambda \bar{w}_0 + (1-\lambda)\bar{w}' \text{ has } \|\bar{w} - \bar{w}_0\| \leq \epsilon.$$

↘ $\bar{w}'$ is outside the local neighborhood of $\bar{w}_0$, but $\bar{w}$ is within the same.

f is convex $\Rightarrow$

$$f(\underbrace{\lambda \bar{w}_0 + (1-\lambda)\bar{w}'}_{\bar{w}}) \leq \lambda f(\bar{w}_0) + (1-\lambda) f(\bar{w}') \quad \text{——— (2)}$$

↘ by $(*)$

Combining (1) and (2) $\Rightarrow$

$$f(\bar{w}_0) \leq f(\bar{w}) \leq \lambda f(\bar{w}_0) + (1-\lambda) f(\bar{w}')$$

$$\Rightarrow (1-\lambda) f(\bar{w}_0) \leq (1-\lambda) f(\bar{w}')$$

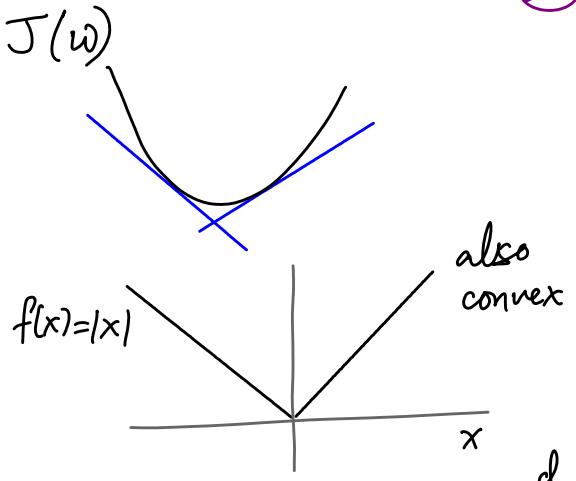
$$\Rightarrow f(\bar{w}') \geq f(\bar{w}_0)$$

$\Rightarrow \bar{w}_0$ is a global minimum.

□

An Observation

A convex function f lies above any tangent to f. We can use this observation to derive a characterization of convex functions in terms of ∇f.



Lemma 4 A differentiable function $f(\bar{w}): \Omega \to \mathbb{R}$ for convex $\Omega \subseteq \mathbb{R}^d$ is a convex function if and only if

$$f(\bar{w}) \geq f(\bar{w}_0) + [\nabla f(\bar{w}_0)]^T (\bar{w} - \bar{w}_0) \quad \text{——— (I)}$$

$\forall\, \bar{w}, \bar{w}_0 \in \Omega$

↪ first derivative condition of convexity

Proof ($\Rightarrow$) Assume f is convex $\Rightarrow$

$\forall\, \bar{w}, \bar{w}_0 \in \Omega$, $\lambda \in [0,1]$

$$f((1-\lambda)\bar{w}_0 + \lambda\bar{w}) \leq (1-\lambda) f(\bar{w}_0) + \lambda f(\bar{w})$$

↪ same as (*), but with λ and $1-\lambda$ flipped

$$\Rightarrow f(\bar{w}_0 + \lambda(\bar{w} - \bar{w}_0)) - f(\bar{w}_0) \leq \lambda\,(f(\bar{w}) - f(\bar{w}_0))$$

Hence, for $\lambda > 0$, we get

$$\frac{f(\bar{w}_0 + \lambda(\bar{w} - \bar{w}_0)) - f(\bar{w}_0)}{\lambda} \leq f(\bar{w}) - f(\bar{w}_0).$$

Taking the limit of LHS as $\lambda \to 0^+$ gives the directional derivative of f at $\bar{w}_0$ in the direction of $\bar{w} - \bar{w}_0$.

$$\Rightarrow [\nabla f(\bar{w}_0)]^T (\bar{w} - \bar{w}_0) \leq f(\bar{w}) - f(\bar{w}_0)$$

$$\Rightarrow f(\bar{w}) \geq f(\bar{w}_0) + \nabla f(\bar{w}_0)^T (\bar{w} - \bar{w}_0)$$

($\Leftarrow$) proof: you can do it yourselves!

□

Note: This definition assumes that f is differentiable. → $f(x)=|x|$ is not differentiable, but is convex!

But still, going one step further, we can give a characterization based on the second derivative of f.

Lemma 5 A twice differentiable function $f: \Omega \to \mathbb{R}$ for convex $\Omega \subseteq \mathbb{R}^d$ is convex iff

$$Hf(\bar{w}) \succeq 0 \quad \forall \bar{w} \in \Omega.$$

→ second derivative condition of convexity

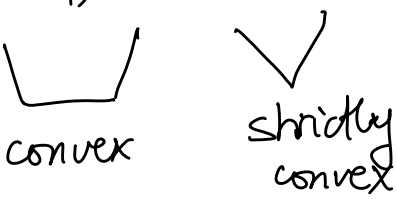
Note: Lemma 4 says that if $\nabla f(\bar{w}_0) = \bar{0}$, then $f(\bar{w}) \geq f(\bar{w}_0) \; \forall \bar{w}$, directly implying that $\bar{w}_0$ is a global optimum.

Strict Convexity

→ a stronger notion of convexity

Def $f: \Omega \to \mathbb{R}$ for convex $\Omega \subseteq \mathbb{R}^d$ is strictly convex if

$$f(\lambda \bar{w}_1 + (1-\lambda)\bar{w}_2) < \lambda f(\bar{w}_1) + (1-\lambda) f(\bar{w}_2) \quad \forall \bar{w}_1, \bar{w}_2 \in \Omega, \lambda \in [0,1].$$



Lemma 6 A strictly convex function can contain at most one critical point. If such a critical point exists, it is the global minimum of the function.

$y = e^x$: strictly convex but has no critical point.

Lemma 7 Let $f: \Omega \to \mathbb{R}$ be convex and $g: \Omega \to \mathbb{R}$ be strictly convex for convex $\Omega \subseteq \mathbb{R}^d$. Then $h: \Omega \to \mathbb{R}$ defined as $h = f + g$ is strictly convex.

This property is used in many ML settings — we add a strictly convex term to convex loss J to make it strictly convex.

For instance, Tikhonov regularization:

$$J = \frac{1}{2}\|D\bar{w} - \bar{y}\|^2 + \frac{1}{2}\|\bar{w}\|^2$$

($\|D\bar{w} - \bar{y}\|^2$: convex; $\|\bar{w}\|^2$: strictly convex)

→ J is strictly convex.

Details of Gradient descent

$$\bar{w} \leftarrow \bar{w} - \alpha \nabla J(\bar{w})$$

α: learning rate; $\nabla J(\bar{w})$: gradient

We consider both these components in detail.

How to compute ∇J?

* analytically (when simple/possible) and hard code it into the computation
  - automatic differentiation (in NNs)..
    → we'll talk about it later...

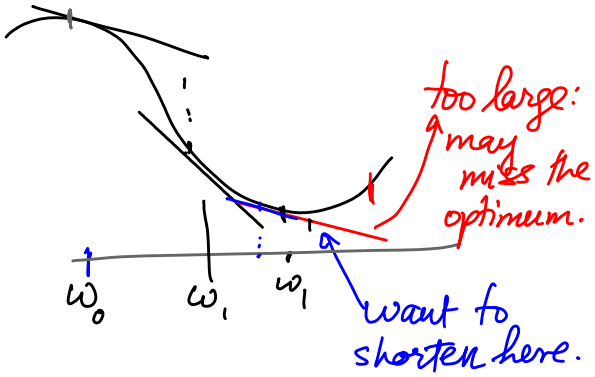
* finite difference approximation: $\nabla J = \begin{bmatrix} \vdots \\ \frac{\partial J}{\partial w_i} \\ \vdots \end{bmatrix}$

We compute $\frac{\partial J}{\partial w_i} \approx \frac{J(w_1, \ldots, w_i + \Delta, \ldots, w_d) - J(\bar{w})}{\Delta_i}$ for "small" Δ

  - can do this calculation for only a subset of $i \in \{1, \ldots, d\}$, especially when $d >> 1$.
    ↪ when d is in the billions, we cannot afford to take the partial derivative in each dimension.
  - We could also use different Δ for each i, (as long as Δ_i is "small").

Changing α

Intuitively, we want to start with a large α so that J can decrease rapidly at start. And then make α smaller when we are "close to" a critical point.
(so, we "decay" α)



$$\bar{w}_{t+1} = \bar{w}_t - \alpha_t \nabla J(\bar{w}_t)$$

t: time ($\equiv$ iteration k as used before)

1. Exponential decay

$$\alpha_t = \alpha_0 e^{(-\mu t)}$$ → controls the rate of decay

2. inverse decay:

$$\alpha_t = \frac{\alpha_0}{1+\mu t}$$

3. Step decay: → decrease α_t every m steps

$$\alpha_t = \frac{\alpha_{t-m+1}}{\mu} \quad \text{when} \quad t = zm \text{ for } z \in \mathbb{N}.$$

4. Bold driver algorithm:

α_0: (initial value)

step t: $\alpha_t = \alpha_{t-1} \times 1.05$ → increase α_t by 5% to keep the decrease of J going

compute J_t

if $J_t < J_{t-1}$
  proceed;
else
  $\alpha_t = \alpha_t/2$
  compute J_t again
end