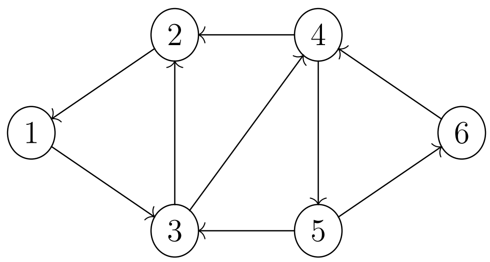
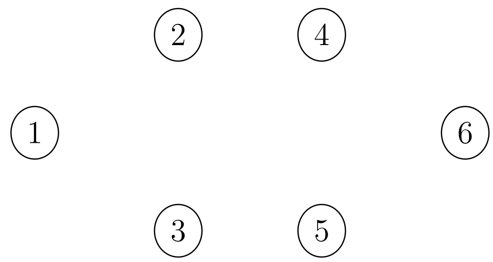
Network Optimization (Fall 2024): Midterm Exam

Name: ________________________________

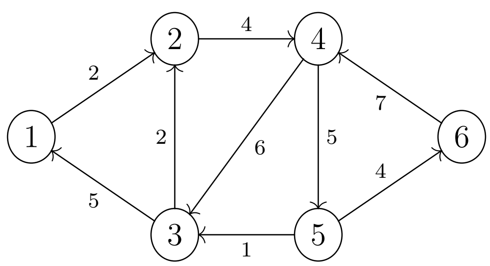
- There are **eight** problems on **four** pages.
- The total points (given in parentheses) add up to 105. You will be graded for 100 points.

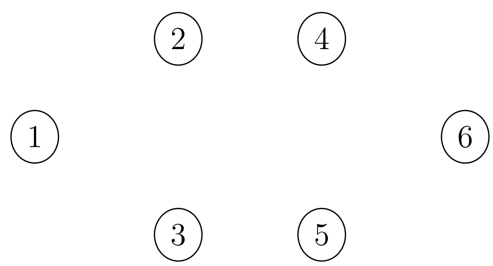
1. (12) Identify all directed cycles in the following network. Make the graph acyclic by removing the *minimum* number of arcs. Give a topological ordering for the resulting acyclic network. You may use the extra copy of the graph to show the cycles or the topological order.



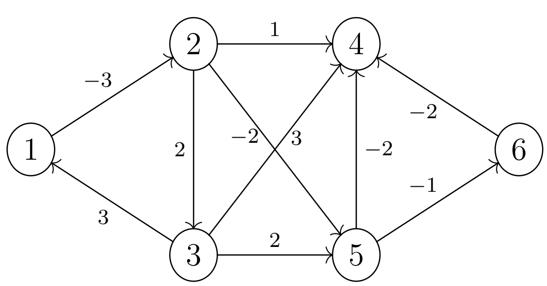


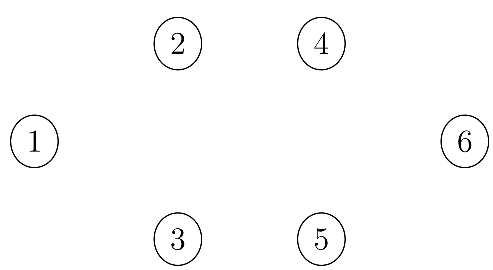
2. (12) Give a decomposition of the flow (the number on each arc (i, j) is the flow x_{ij}) into flows along paths and cycles. Is this decomposition unique?





3. (12) Find the shortest path from node $s = 3$ to every other node in the graph (the number on arc (i, j) is c_{ij}). Show the optimal shortest path distance labels and the shortest path tree.





4. (14) Consider the min-cost flow problem where there are lower and upper bounds on the total flow in to each node given by γ_ℓ and γ_u, and lower and upper bounds on the total flow out of each node given by δ_ℓ and δ_u. Explain how you will transform this problem into an equivalent min-cost flow problem in the standard form.

5. (14) For a network $G = (N, A)$, assume that $c_{ij} \geq 0 \; \forall \; (i,j) \in A$. Describe an algorithm to find a shortest walk from node s to node t that visits a specific node p. What is the complexity of your algorithm? Will this walk always be a *path*?

6. (12) In each part, choose the best option. Give a **brief** justification.

   (a) Let $\ell_{\text{bfs}}(i), \ell_{\text{dfs}}(i)$, and $\ell_{\text{sp}}(i)$ denote the number of arcs in the path from node s to node i in the breadth-first search tree, the depth-first search tree, and the shortest path tree of a network $G = (N, A)$, respectively. Also define $\ell_{\text{s}}(i) = \min\{\ell_{\text{bfs}}(i), \ell_{\text{dfs}}(i)\}$.

   i. $\ell_{\text{s}}(i) \geq \ell_{\text{sp}}(i) \;\; \forall \, i \in N$.
   ii. $\ell_{\text{s}}(i) \leq \ell_{\text{sp}}(i) \;\; \forall \, i \in N$.
   iii. $\ell_{\text{s}}(i) \leq \ell_{\text{sp}}(i)$ when $c_{ij} \geq 0 \; \forall \; (i,j) \in A$.
   iv. none of the above.

   (b) Let $d(i)$ represent the shortest path distances from node s to i in a graph $G = (N, A)$. Suppose the shortest path from s to a node k has exactly three arcs. Now, obtain another graph $G' = (N, A)$ by setting $c'_{ij} = c_{ij} + 10$ for all $(i,j) \in A$. Let $d'(i)$ represent the shortest path distances from s to node i in the graph G'. Then

   i. $d'(k) = d(k) + 30$
   ii. $d'(k) \leq d(k) + 30$
   iii. $d'(k) \geq d(k) + 30$
   iv. there is not enough information

7. (14) For each part, answer TRUE or FALSE. **Justify** your answer.

   (a) An arc is *vital* if its removal from the network increases the shortest path distance from node s to node t. A *most vital arc* is a vital arc whose removal produces the maximum increase in the s-t shortest path distance. We can find a most vital arc in a network in $O(nT)$ time, where T is the time needed to solve a shortest path problem on the network.

   (b) $n^{\log \log n} = \Theta(n \log n)$.

8. (15) In each of t periods, Butters can buy, sell, or hold for later sale some commodity. In period i, he can buy at most α_i units, can holdover at most β_i units for the next period, and must sell at least γ_i units. Let c_i, h_i, and p_i denote the per unit buying cost, holding cost, and selling price in period i, respectively. Formulate the problem of determining the optimal buy-hold-sell policy (which maximizes overall profit in t periods) as a network flow problem.