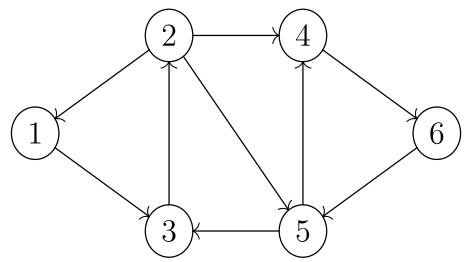
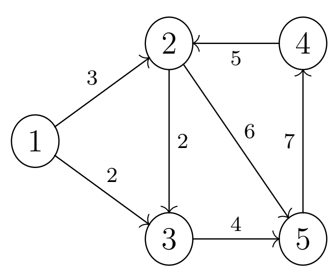
Network Optimization (Fall 2024): Practice Midterm Exam

- The total points (given in parentheses) add up to 105. You will be graded for 100 points.
- Try to give **brief** answers.
- *"Time is precious."* – anonymous.

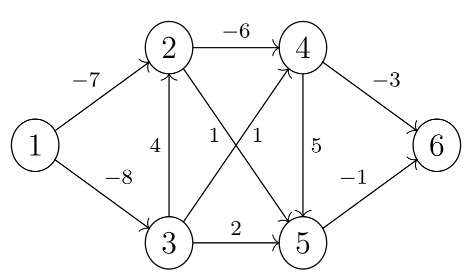
1. (10) Give a BFS tree and a DFS tree for the following graph with starting node $s = 2$.



2. (15) Give a decomposition of the flow (the number on each arc (i, j) is the flow x_{ij}) into flows along paths and cycles. Is this decomposition unique?



3. (15) Find the *longest path* from node $s = 1$ to every other node in the graph (the number on arc (i, j) is c_{ij}). Show the optimal longest path distance labels and the longest path tree.



4. (12) Suppose you are given a min-cost flow problem on a network $G = (N, A)$ with $|N| = n,\ l_{ij} = 0\ \forall\,(i,j) \in A$, and $0 < u_{ij} < \infty\ \forall\,(i,j) \in A$. Further, the arcs going out of each supply node have enough capacity to carry all the supply from that node out, and the arcs coming in to each demand node have enough capacity to get flow in to that node to satisfy all its demand. In other words,

$$\sum_{(i,j)\in A} u_{ij} \geq b(i)\ \forall\, i \in N \text{ with } b(i) > 0, \text{ and } \sum_{(j,i)\in A} u_{ji} \geq -b(i)\ \forall\, i \in N \text{ with } b(i) < 0.$$

Describe how you can transform this min-cost flow problem to an equivalent *transportation problem*.

5. (12) In each part, choose the best option. Give a **brief** justification.

   (a) Let $\ell_{\text{DFS}}(i)$ and $\ell_{\text{SP}}(i)$ denote the number of arcs in the path from node s to node i in the depth-first search (DFS) tree and the shortest path (SP) tree of a network $G = (N, A)$, respectively.

      i. $\ell_{\text{DFS}}(i) \geq \ell_{\text{SP}}(i)\ \ \forall\, i \in N$.
      ii. $\ell_{\text{DFS}}(i) \leq \ell_{\text{SP}}(i)\ \ \forall\, i \in N$.
      iii. $\ell_{\text{DFS}}(i) \geq \ell_{\text{SP}}(i)$ when $c_{ij} > 0\ \forall\,(i,j) \in A$.
      iv. none of the above.

   (b) The decomposition of a flow in arcs to flows in paths and cycles

      i. can never be unique.
      ii. is unique if there are only path flows (and no cycle flows) in the decomposition.
      iii. is unique if there are only cycle flows (and no path flows) in the decomposition.
      iv. can be unique depending on the network in question.

6. (16) For each part, answer TRUE or FALSE. **Justify** your answer.

   (a) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

   (b) Let $p(i)$ denote the number of distinct paths from node s to node i in an acyclic graph G. We can find $p(i)$ for all nodes i in $O(m)$ time.

7. (8) Prove the following claim, or give a counterexample. If all the arc costs in a network are relatively prime, then it has a unique shortest path tree.

8. (17) The capacity of a certain facility is to be expanded over n time periods by adding an increment $y_i \in [0, v_i]$ at time period $i = 0, \ldots, n-1$ (v_i's are given scalars). Thus, if x_i is the capacity at the beginning of time period i, then

$$x_{i+1} = x_i + y_i, \ \ i = 0, \ldots, n-1.$$

Given x_0, consider the problem of finding y_i for $i = 0, \ldots, n-1$ such that each x_i lies within the interval $[l_i, u_i]$ for $i = 1, \ldots, n$ (l_i, u_i are given scalars), and the cost

$$\sum_{i=0}^{n-1} (c_{i+1} x_{i+1} + a_i y_i)$$

is minimized for given costs $c_1, \ldots, c_n$ and $a_0, \ldots, a_{n-1}$. Formulate this problem as a network flow problem.