Math 466/566 (Fall 2024): Flow Decomposition Algorithm

INPUT : $G = (N, A)$ with arc flow $\mathbf{x}$ (m-vector of flows x_{ij}).
OUTPUT: $\mathcal{P}$ with $f(P)\ \forall\ P \in \mathcal{P},\ \mathcal{W}$ with $f(W)\ \forall\ W \in \mathcal{W}$.
(here, $\mathcal{P}$ and $\mathcal{W}$ are the sets of paths and cycles, and f is flow).

Notation:

$\mathbf{y}$: intermediate flow
$G(\mathbf{y}) = (N(\mathbf{y}), A(\mathbf{y}))$: graph corresponding to flow $\mathbf{y}$
$A(\mathbf{y}) = \{(i,j) \in A \mid y_{ij} > 0\}$ (arcs with positive flow in $\mathbf{y}$)
$N(\mathbf{y}) = \{i \mid (i,j) \in A(\mathbf{y}) \text{ or } (j,i) \in A(\mathbf{y})\}$ (nodes incident to arcs in $A(\mathbf{y})$)
$\mathcal{S} = \{i \mid b(i) > 0\}$ (supply nodes)
$\mathcal{D} = \{i \mid b(i) < 0\}$ (demand nodes)
for $P \in \mathcal{P},\ \Delta(P) = \min\{\, b(s), -b(t), \min\{y_{ij} \mid (i,j) \in P\}\,\}$ (capacity of path)
for $W \in \mathcal{W},\ \Delta(W) = \min\{y_{ij} \mid (i,j) \in W\}$ (capacity of cycle)

Algorithm

$\mathbf{y} := \mathbf{x}$, $\mathcal{P} = \emptyset$, $\mathcal{W} = \emptyset$, assign $A(\mathbf{y}), N(\mathbf{y}), \mathcal{S},$ and $\mathcal{D}$. (Initialization)
while $\mathbf{y} \neq \mathbf{0}$ **do**
begin
    $s =$ **Select**$(\mathbf{y})$
    Search$(s, \mathbf{y})$
    if cycle W found **then do**
    begin
        $\mathcal{W} = \mathcal{W} \cup \{W\}$
        $f(W) = \Delta(W)$
        $y_{ij} = y_{ij} - \Delta(W)\ \forall\ (i,j) \in W$
        update $A(\mathbf{y}), N(\mathbf{y})$
    end
    if path P found **then do**
    begin
        $\mathcal{P} = \mathcal{P} \cup \{P\}$
        $f(P) = \Delta(P)$
        $y_{ij} = y_{ij} - \Delta(P)\ \forall\ (i,j) \in P$
        $b(s) = b(s) - \Delta(P)$ (s is starting node of P)
        $b(t) = b(t) + \Delta(P)$ (t is ending node of P)
        update $A(\mathbf{y}), N(\mathbf{y}), \mathcal{S}, \mathcal{D}$
    end
end

Select$(\mathbf{y})$
    if $\mathcal{S} \neq \emptyset$ **then** choose $s \in \mathcal{S}$;
    else choose $s \in N(\mathbf{y})$;

Search$(s, \mathbf{y})$
    Do DFS starting with node s until finding a cycle W in $G(\mathbf{y})$
    or a path P in $G(\mathbf{y})$ from node s to a node $t \in \mathcal{D}$