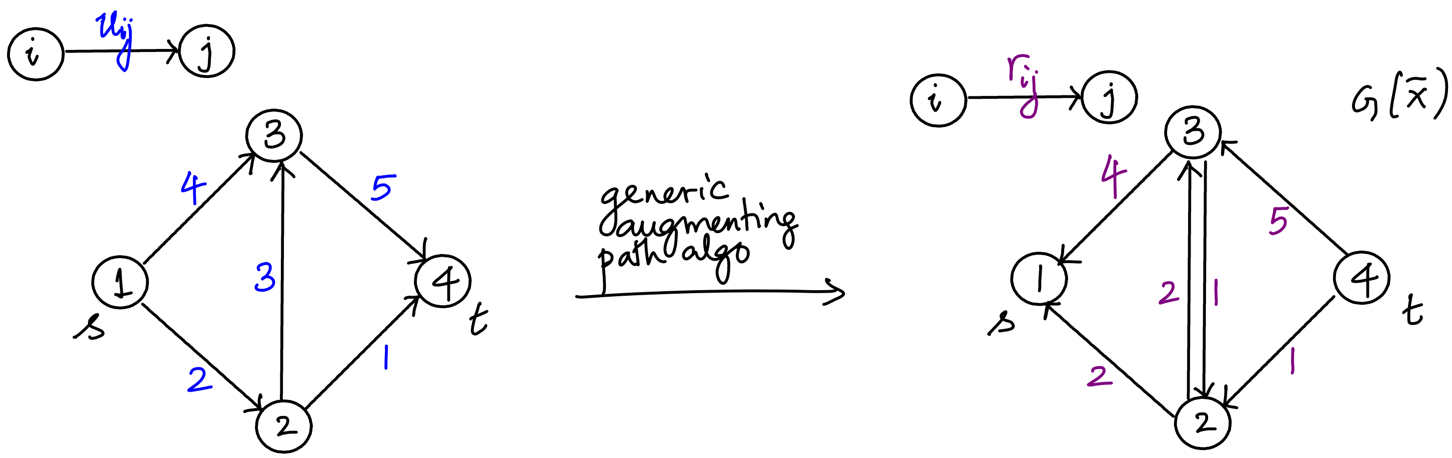
MATH 566: Lecture 17 (10/15/2024)

Today: * finiteness of augmenting path algo
* max-flow min-cut theorem (MFMC)

To finish the example on augmenting path algorithm, we consider the following:
How to obtain optimal x_{ij}'s from final $G(\bar{x})$ once the algorithm terminates?

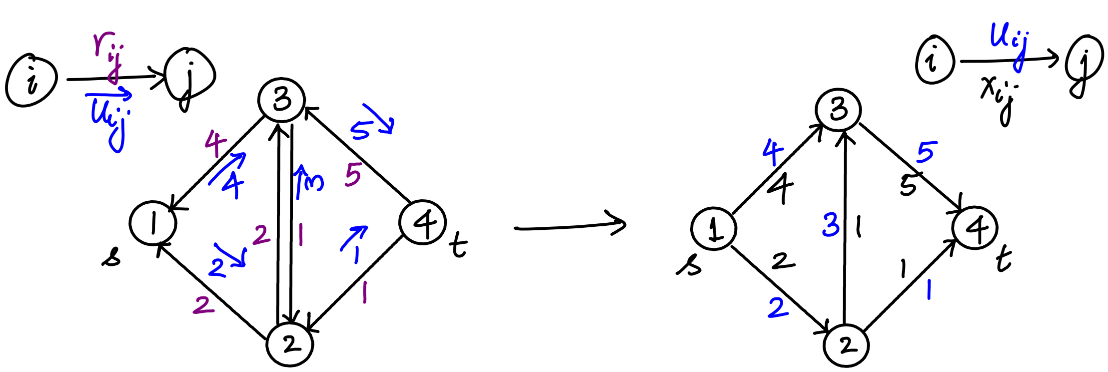


Recall : $r_{ij} = u_{ij} - x_{ij} + x_{ji}$

$$\Rightarrow x_{ij} - x_{ji} = u_{ij} - r_{ij}$$

If $u_{ij} \geq r_{ij}$, set $x_{ij} = u_{ij} - r_{ij}$, $x_{ji} = 0$;
else set $x_{ij} = 0$ and $x_{ji} = r_{ij} - u_{ij}$.
$(u_{ij} < r_{ij})$

Indeed, only one of x_{ij} and x_{ji} will be >0, not both!



Notice we obtain $x_{23} = 1$ from $r_{23} = 2$ or, equivalently from $r_{32} = 1$.

Proof of Finiteness of the Generic Augmenting Path algorithm

We first prove finiteness of the generic algorithm, and then consider details of implementation (to get polynomial time implementations).

Recall, we assume $u_{ij} \in \mathbb{Z}_{\geq 0}$ (non negative integers). We now assume all u_{ij}'s are also finite. → The more general case when some u_{ij}'s are ∞ can be handled similarly.

1. **Lemma** The residual capacities (r_{ij}) are integral after each iteration.
2. The capacity of each augmenting path is at least 1 (as all $r_{ij} \geq 1$).
3. Augmenting along a path P decreases r_{si} for some arc (s,i) by at least one unit.
4. Augmentation never increases r_{si} for any i.
5. $\sum'_{(s,i)\in A} r_{si}$ keeps decreasing, and is lower bounded by zero. $\left(\sum'_{(s,i)\in A} r_{si} \geq 0\right)$.

$\Rightarrow$ The # augmentations $= \underbrace{O(nU)}_{\text{finite!}}$ where $U = \max_{(i,j)\in A} \{u_{ij}\}$.

In fact, we can say the # augmentations $= O(nU_s)$, where $U_s = \max_{(s,i)\in A} \{u_{si}\}$.

So, we could have an arc with a much larger capacity further down the network (i.e., farther away from s). But since we are sending flow out of s, we may not be able to use this high-capacity arc to its full.

How do we know when a flow $\bar{x}$ is optimal?

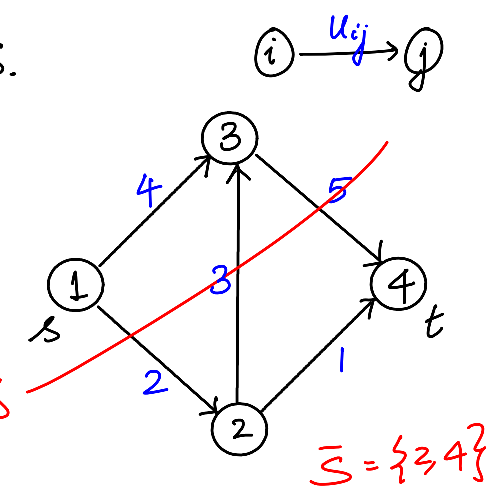
1. No augmenting path in $G(\bar{x})$.
2. Max-flow Min-Cut (MFMC) theorem (duality)

} Max-flow optimality conditions

Def An **s-t cut** is a partition of node set N into two disjoint sets $S, \bar{S}$ such that $s \in S$, $t \in \bar{S}$. $\Rightarrow (S \cup \bar{S} = N,\ S \cap \bar{S} = \phi)$

A **forward arc** is $(i,j) \in A$ with $i \in S$, $j \in \bar{S}$.

A **backward arc** is $(i,j) \in A$ with $i \in \bar{S}$, $j \in S$.



Example For $S = \{1,3\}$, $\bar{S} = \{2,4\}$, (1,2) and (3,4) are forward arcs, and (2,3) is a backward arc.

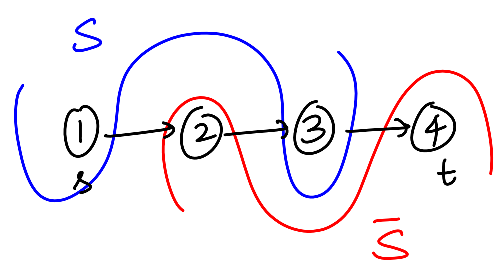
Def The **capacity** of an s-t cut $[S, \bar{S}]$ is

$$u[S,\bar{S}] = \sum_{\substack{i \in S, j \in \bar{S} \\ (i,j) \in A}} u_{ij} \qquad (1)$$

$\rightarrow$ sum of all forward arc capacities

e.g., $u[S,\bar{S}] = u_{12} + u_{34} = 2 + 5 = 7$, above

We are not assuming anything about the connectivity of subgraphs defined by S and $\bar{S}$, e.g., the following scenario is perfectly valid:



Property 6.1 The value v of any feasible flow $\bar{x}$ is at most $u[S,\bar{S}]$ of any s-t cut $[S,\bar{S}]$.

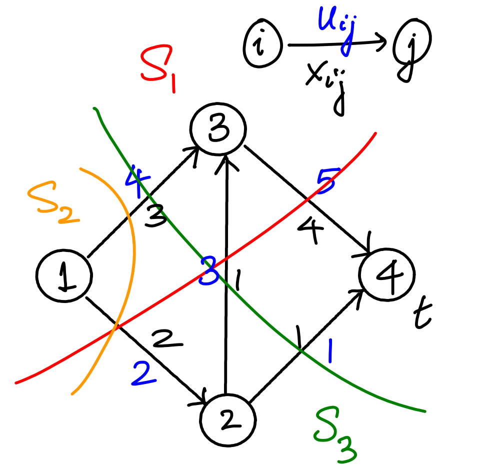
We present two claims, from which the above Property follows.

Def The **flow across a cut** $[S,\bar{S}]$ is

$$F_{\bar{x}}[S,\bar{S}] = \sum_{i\in S, j\in\bar{S}} x_{ij} - \sum_{i\in S, j\in \bar{S}} x_{ji}. \qquad (2)$$

$S_1 = \{1,3\}$

$$F_{\bar{x}}[S_1,\bar{S}_1] = \underset{x_{12}+x_{34}-x_{23}}{2+4-1} = 5$$



$S_2 = \{1\}$

$$F_{\bar{x}}[S_2,\bar{S}_2] = \underset{x_{13}+x_{12}}{3+2} = 5$$

$S_3 = \{1,2\}$

$$F_{\bar{x}}[S_3,\bar{S}_3] = \underset{x_{13}+x_{23}+x_{24}}{3+1+1} = 5$$

Notice $F_{\bar{x}}[S,\bar{S}]$ = value of flow in each case. Indeed, this is not a coincidence!

Claim 1 If $[S,\bar{S}]$ is an s-t cut, then $F_{\bar{x}}[S,\bar{S}] = v$.

Proof Add flow balance equations for all $i\in S$.

Recall, $\sum_{(s,j)\in A} x_{sj} - \sum_{(j,s)\in A} x_{js} = v$ is the equation for $i=s$. Overall, we will be left with flows x_{ij} and $-x_{ji}$ for $i\in S, j\in\bar{S}$, and the flows within S will cancel in $\pm$ pairs. □

Claim 2 $F_{\bar{x}}[S,\bar{S}] \le u[S,\bar{S}]$ for any s-t cut $[S,\bar{S}]$.

Proof For $i \in S$, $j \in \bar{S}$, $x_{ij} \le u_{ij}$, and $x_{ji} \ge 0$ hold.

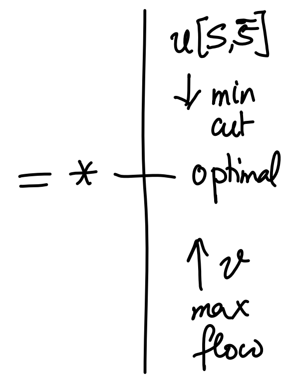
so (2) $\Rightarrow$ $F_{\bar{x}}[S,\bar{S}] = \sum' x_{ij} - \sum' x_{ji} \le \sum_{i\in S, j \in \bar{S}} u_{ij} - \sum' 0$

$$= u[S,\bar{S}], \text{ by } (1). \qquad \square$$

Note that Property 6.1 follows directly from Claims 1 and 2 above.

The **Max-Flow Min-Cut** Theorem (MFMC)

We state a more general theorem, from which MFMC follows as a corollary.



Theorem 6.3 (Optimality Conditions for Max flow)

The following statements are equivalent.

(1) A flow $\bar{x}$ is maximum. the m-vector of arc flows x_{ij}

(2) There is no augmenting path in $G(\bar{x})$.

(3) There is an s-t cut $[S,\bar{S}]$ whose capacity is equal to the value of the flow $\bar{x}$, i.e., $u[S,\bar{S}] = v$.

Corollary 6.4 (MFMC Theorem) The maximum flow value is equal to the minimum capacity of an s-t cut.

Proof (of Theorem 6.3) (1) $\Rightarrow$ (2). We argue $\neg(2) \Rightarrow \neg(1)$

"negation" or "not"

If there is an augmenting path in $G(\bar{x})$, $\bar{x}$ is not maximum.

(3) $\Rightarrow$ (1)

By (3), we have $v = u[S, \bar{S}]$.

By Claim (1), $v = F_{\bar{x}}[S, \bar{S}]$. Hence $F_{\bar{x}}[S,\bar{S}] = u[S,\bar{S}]$.

By Claim (2), $F_{\bar{x}}[S,\bar{S}] \le u[S,\bar{S}]$. Hence we have optimality.

(2) $\Rightarrow$ (3) There is no augmenting path in $G(\bar{x})$.

Let $S = \{ i \in N \mid i \text{ is reachable from } s \text{ in } G(\bar{x}) \}$, and

$\bar{S} = N \setminus S$.

$\Rightarrow$ There is no arc in $G(\bar{x})$ from S to $\bar{S}$.

forward arcs are all full, and backward arcs have no flow.

So, $i \in S, j \in \bar{S}$ $(r_{ij} = 0) \Rightarrow x_{ij} = u_{ij}$, and

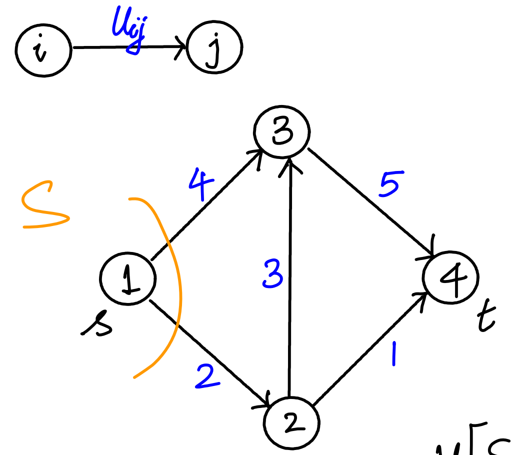
$i \in S, j \in \bar{S}$ $(r_{ji} = u_{ji}) \Rightarrow x_{ji} = 0$.

$$\Rightarrow F_{\bar{x}}[S,\bar{S}] = \sum_{i \in S, j \in \bar{S}} x_{ij} - \sum_{i \in S, j \in \bar{S}} x_{ji}$$

$$= \sum_{i \in S, j \in \bar{S}} u_{ij} - \sum_{i \in S, j \in \bar{S}} 0 = u[S,\bar{S}].$$

$\square$

Property To obtain a min cut from a max flow $\bar{x}$, set $S = \{\text{all nodes reachable from } s \text{ in } G(\bar{x})\}$.



$$u[S, \bar{S}] = 4 + 2 = 6 = v$$