MATH 566 : Lecture 24 (11/07/2024)

Today:
* reduced cost optimality conditions
* complementary slackness conditions (CSCs)
* successive SP algo — set up

2. Reduced Cost Optimality Conditions

$c_{ij}^{\bar{\pi}} = c_{ij} - \pi(i) + \pi(j)$

In SP, $c_{ij}^{d} \geq 0 \iff d(j) \leq d(i) + c_{ij}$ are the optimality conditions.

AMO Property 9.2

(a) For a path P from node k to l,

$$\sum_{(i,j)\in P} c_{ij}^{\bar{\pi}} = \sum_{(i,j)\in P} c_{ij} - \underbrace{\pi(k) + \pi(l)}_{\text{fixed for a given } \bar{\pi}}.$$

(b) For a directed cycle W, $\displaystyle\sum_{(i,j)\in W} c_{ij}^{\bar{\pi}} = \sum_{(i,j)\in W} c_{ij}$.

Implications

1. A given set of node potentials $\bar{\pi}$ does not change the shortest path from any node k to node l.
2. A directed negative cycle W w.r.t c_{ij} is also a directed negative cycle w.r.t $c_{ij}^{\bar{\pi}}$.

What about the total cost for the MCF problem (under reduced costs)?

With total cost $\bar{c}^T\bar{x} = \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij}$ and $\bar{c}^{\bar{\pi}T}\bar{x} = \displaystyle\sum_{(i,j)\in A} c_{ij}^{\bar{\pi}}x_{ij}$,

we would like to get $\bar{c}^T\bar{x} - \bar{c}^{\bar{\pi}T}\bar{x} = \text{constant}$.

$$
\begin{aligned}
\bar{c}^T\bar{x} - \bar{c}^{\bar{\pi}^T}\bar{x} &= \sum_{(i,j)\in A}{}' \left(c_{ij} - c_{ij}^{\bar{\pi}}\right) x_{ij} \\
&= \sum_{(i,j)\in A}{}' \left(\pi(i) - \pi(j)\right) x_{ij} \\
&= \sum_{i\in N}{}' \pi(i) \underbrace{\left(\sum_{(i,j)\in A}{}' x_{ij} - \sum_{(j,i)\in A}{}' x_{ji}\right)}_{b(i)} \\
&= \sum_{i\in N}{}' \pi(i)\cdot b(i), \quad \text{a constant for a given } \bar{\pi}.
\end{aligned}
$$

Hence the optimal solutions w.r.t c_{ij} remain optimal w.r.t. $c_{ij}^{\bar{\pi}}$.

Reduced Cost Optimality Conditions: A feasible flow $\bar{x}$ is an optimal solution for the MCF problem iff <u>some</u> node potentials $\bar{\pi}$ satisfy

$$c_{ij}^{\bar{\pi}} \geqslant 0 \quad \forall (i,j) \in G(\bar{x})$$

there exists a set of node potentials $\bar{\pi}$ satisfying $c_{ij}^{\bar{\pi}} \geqslant 0$.

We now show that the reduced cost optimality conditions are equivalent to the negative cycle optimality conditions.

Proof

IDEA: No negative cycle in $G(\bar{x}) \iff c_{ij}^{\bar{\pi}} \geq 0 \ \forall (i,j) \in G(\bar{x})$.

($\Leftarrow$) Let $c_{ij}^{\bar{\pi}} \geq 0 \ \forall (i,j) \in G(\bar{x})$.

$\Rightarrow$ For every cycle W in $G(\bar{x})$, $\sum_{(i,j)\in W} c_{ij}^{\bar{\pi}} = \sum_{(i,j)\in W} c_{ij} \geq 0$.

$\Rightarrow$ There is no negative cycle in $G(\bar{x})$.

($\Rightarrow$) $G(\bar{x})$ has no negative cycle.

We want to show $c_{ij}^{\bar{\pi}} \geq 0 \ \forall (i,j) \in G(\bar{x})$ for some $\bar{\pi}$. Indeed, we can use $\pi(i) = -d(i)$, where $d(i)$ = shortest path distance from node 1 to node i, $\forall i \in N$.

could use any node

Since $c_{ij}^{\bar{\pi}} = c_{ij} + d(i) - d(j) \geq 0$, we have optimality. $\square$

Economic Interpretation of SP optimality Conditions:

Think of c_{ij} = cost of transporting 1 unit from i to j, and $d(i) = -\pi(i)$ = cost of buying 1 unit at node i. Then $d(j) \leq d(i) + c_{ij}$ says that the cost of buying 1 unit at node j is no more than the cost to buy it at node i and transporting it to node j.

3. Complementary Slackness Optimality Conditions

Let $\bar{x}$ be a feasible flow and $\bar{\pi}$ be a set of node potentials. Then $(\bar{x}, \bar{\pi})$ is optimal iff the following conditions hold.

1. If $c_{ij}^{\bar{\pi}} > 0$ then $x_{ij} = 0$.
2. If $c_{ij}^{\bar{\pi}} < 0$ then $x_{ij} = u_{ij}$.
3. If $0 < x_{ij} < u_{ij}$ then $c_{ij}^{\bar{\pi}} = 0$.

Note

* The CSCs are specified on G, and not on $G(\bar{x})$.
  Recall that the negative cycle and reduced cost optimality conditions are specified on $G(\bar{x})$.

* Condition 3 does not imply the reverse result, i.e., $c_{ij}^{\bar{\pi}} = 0$ does **not** imply $0 < x_{ij} < u_{ij}$.

Intuitively, $c_{ij}^{\bar{\pi}} = 0$ means the flow along (i,j) does not cost anything. Hence it could be any value — in particular, it could be at its lower or upper bound, or in between.

Proof

Show reduced cost optimality conditions $\iff$ CSCs.

We show that reduced cost optimality conditions imply CSCs.

1. If $c_{ij}^{\bar{\pi}} > 0 \implies r_{ji} = 0$ (or $(j,i) \notin G(\bar{x})$), → assuming reduced cost optimality conditions

as $c_{ji}^{\bar{\pi}} = -c_{ij}^{\bar{\pi}} < 0$ will violate reduced cost optimality conditions.

But $r_{ji} = 0 \implies x_{ij} = 0$.

2. If $c_{ij}^{\bar{\pi}} < 0 \implies r_{ij} = 0 \implies x_{ij} = u_{ij}$. → violates reduced cost optimality conditions, unless $(i,j) \notin G(\bar{x})$

3. If $0 < x_{ij} < u_{ij}$ then $r_{ij} > 0$ and $r_{ji} > 0$.

i.e., both (i,j) and $(j,i) \in G(\bar{x})$.

$\implies c_{ij}^{\bar{\pi}} \geq 0$ and $c_{ji}^{\bar{\pi}} = -c_{ij}^{\bar{\pi}} \geq 0$ according to the reduced cost optimality conditions.

$\implies c_{ij}^{\bar{\pi}} = 0$.

Successive Shortest Path (SSP) Algorithm for MCF

IDEA Combine tools froms Preflow-push and SAP, apply to MCF now.

* Maintain optimality in terms of reduced cost optimality conditions.
* Strive for feasibility (for flow-balance; bounds hold always).

We present some definitions first.

Def A **pseudoflow** is $\bar{x} \in \mathbb{R}^m$ that satisfies bounds in the MCF formulation, i.e., $0 \le x_{ij} \le u_{ij}$, but not necessarily the flow balance constraints.

Def The **imbalance** at node i is

$$e(i) = \underbrace{b(i)}_{\text{supply/demand}} + \underbrace{\sum_{(j,i)\in A}' x_{ji}}_{\text{inflow}} - \underbrace{\sum_{(i,j)\in A}' x_{ij}}_{\text{outflow}}$$

recall the definition of excess nodes in preflow push for max flow

If $e(i) > 0$, i is an **excess** node, and it belongs to set E, and if $e(i) < 0$, i is a **deficit** node, and it belongs to set D.

Note that $\sum_{i\in N}' e(i) = \sum_{i \in N}' b(i) = 0$, and

$$\sum_{i\in E}' e(i) = -\sum_{j \in D}' e(j).$$

Hence, if G has an excess node, it must have a deficit node.

SSP Algo: Maintain optimality as $c_{ij}^{\bar{\pi}} \geq 0 \; \forall \, (i,j) \in G(\bar{x})$, and strive for feasibility, i.e., $e(i) = 0 \; \forall \, i \in N$.

We present a key lemma first, which is used in the SSP algorithm.

AMO Lemma 9.11

Let $\bar{x}$ be a pseudoflow, $\bar{\pi}$ a set of node potentials satisfying $c_{ij}^{\bar{\pi}} \geq 0 \; \forall \, (i,j) \in G(\bar{x})$. Let $\bar{d}$ be the set of SP distances from some node s in $G(\bar{x})$ to all (other) nodes in $G(\bar{x})$ using $c_{ij}^{\bar{\pi}}$ as arc lengths. The following results hold.

(a) Let $\bar{\pi}' = \bar{\pi} - \bar{d}$. Then $c_{ij}^{\bar{\pi}'} \geq 0 \; \forall \, (i,j) \in G(\bar{x})$; and

(b) $c_{ij}^{\bar{\pi}'} = 0 \; \forall \, (i,j)$ in a shortest path from s to another node in $G(\bar{x})$.

Proof

(a) $\bar{d}$: SP distances w.r.t. $c_{ij}^{\bar{\pi}}$ in $G(\bar{x})$

$$\Rightarrow \quad d(j) \leq d(i) + c_{ij}^{\bar{\pi}} \; \forall \, (i,j) \in G(\bar{x}).$$

Using $c_{ij}^{\bar{\pi}} = c_{ij} - \pi(i) + \pi(j)$ gives

$$d(j) \leq d(i) + c_{ij} - \pi(i) + \pi(j)$$

$$\Rightarrow \quad c_{ij} - [\pi(i) - d(i)] + [\pi(j) - d(j)] \geq 0$$

$$\Rightarrow \quad c_{ij}^{\bar{\pi}'} \geq 0 \quad \forall \, (i,j) \in G(\bar{x}), \text{ where } \bar{\pi}' = \bar{\pi} - \bar{d}.$$

Thus, decreasing node potentials $\bar{\pi}$ by SP distances $\bar{d}$ in $G(\bar{x})$ maintains reduced cost optimality.

(b) Along the SP tree in $G(\bar{x})$, $d(j) = d(i) + c_{ij}^{\bar{\pi}}$.

$$\Rightarrow \quad c_{ij}^{\bar{\pi}'} = 0 \quad \forall (i,j) \text{ in the SP tree.}$$

$\square$