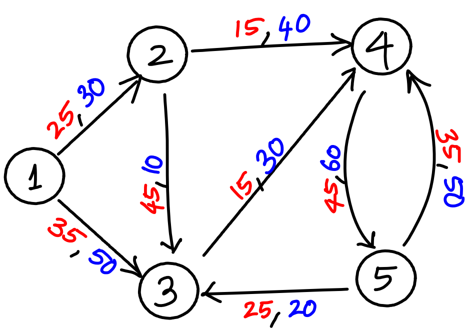
MATH 566: Lecture 5 (09/03/2024)

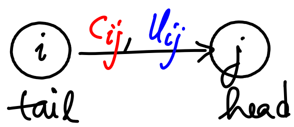
Today: * forward star representation
* network transformations

Forward Star Representation

A list of info about arcs, set as a table. There is one row for each arc $(i,j) \in A$, and 3–5 columns listing the tail (i), head (j), and c_{ij}, l_{ij}, and u_{ij} values.

Illustration





T	H	COST	UB
1	2	25	30
1	3	35	50
2	3	45	10
2	4	15	40
3	4	15	30
4	5	45	60
5	3	25	20
5	4	35	50

Assume $l_{ij} = 0 \;\; \forall (i,j) \in A$

The main matrix is $m \times 3$, or $m\times 4$ or $m \times 5$, depending on which of the parameters c_{ij}, l_{ij}, and u_{ij} are given. The $b(i)$ values are stored in a separate n-vector.

lexicographic order

Notice that the information is stored in the order of the arcs. To extract the outarc list efficiently from this matrix notation, we use one more vector named the **point**. Most algorithms run steps of operations on the outarcs or inarcs of each node. Hence it is important to have these lists readily available, or easily extractable.

	T	H	COST	UB
→ 1	1	2	25	30
2	1	3	35	50
→ 3	2	3	45	10
4	2	4	15	40
→ 5	3	4	15	30
→ 6	4	5	45	60
→ 7	5	3	25	20
8	5	4	35	50

point(i)

	point(i)
1	1
2	3
3	5
4	6
5	7
$(n+1) \to 6$	9 ← $m+1$ (#arcs $+1$)

$$\text{point}(n+1) = m+1 \quad \text{(always)}$$

For $1 \le i \le n$, point(i) stores the row # (index) of the first arc (in lex order) that has i as its tail. Point($n+1$) is always set to $m+1$ (#arcs $+ 1$).

The outarcs of node i are in rows indexed point(i) to point($i+1$)-1, for $1 \le i \le n$.

For instance, for node 2, point(2)$=3$, point($2+1$)$=$ point(3) $= 5$. Hence, the outarcs of node 2 are in rows 3 to $5-1$, i.e., rows 3 and 4.

The matrix ($m \times 3$, $m \times 4$, ...) of arcs list along with the point vector (and $b(\cdot)$ vector) is the **forward star representation** of G.

Note: If node i has no outarcs, we set point(i)$=$ point($i+1$), so that the indices of outarcs of node i is an empty set.

Here is a quick example: we reverse arc (3,4) in the example network. Rest of the network remains unchanged. Note that node 3 now has no outarcs. The corresponding point(·) vector is shown to the right.

	T	H	COST	UB
1	1	2	25	30
2	1	3	35	50
3	2	3	45	10
4	2	4	15	40
5	4	3	15	30
6	4	5	45	60
7	5	3	25	20
m=8	5	4	35	50

point(i)

i	point(i)
1	1
2	3
3	5
4	5
5	7
6	9

Notice how the outarcs of node 2 are still recorded correctly: arcs point(2) = 3 to point(3) − 1 = 4.

Equivalently, we could have a matrix of inarcs along with the associated parameters (c_{ij}, u_{ij}, l_{ij}), and create the **rpoint** vector, which is the ***reverse point*** vector. The in-arcs of node i are precisely the arcs indexed rpoint(i) to rpoint(i+1) − 1.

For small or moderately sized networks, it may be simpler to initialize the A{·} lists as a cell array (outarc list)! For large networks, the use of point is recommended.

Here are some basic Matlab commands and files

netdata.m

```
%% Math 566 (Fall 2024)
%% Matlab code to extract out- and in-arc lists
%% from the forward star representation

%T: TAIL,  H: HEAD
%DEGO: outdegree,  DEGI: indegree
%n: number of nodes,  m: number of arcs
%A{i}:  out-arc list at node i, cell array
%AI{i}: in-arc list at node i, cell array

DEGO=zeros(1,n);
DEGI=DEGO;
A{1,n} = [];
AI{1,n} = [];
for k=1:m
  i=T(k);
  j=H(k);
  pos=DEGO(i)+1;
  DEGO(i)=pos;
  A{i}(pos)=k;

  posI=DEGI(j)+1;
  DEGI(j)=posI;
  AI{j}(posI)=k;
end%for
```

Example1.m

```
%% Math 566 (Fall 2024)
%%
%% Network from Lecture 4, which is a slighty modified
%% version of the network in AMO Figure 2.13, page 32.

data=[1  1 2 25 30
      2  1 3 35 50
      3  2 3 45 10
      4  2 4 15 40
      5  3 4 15 30
      6  4 5 45 60
      7  5 3 25 20
      8  5 4 35 50 ];

% The first column is just the arc index (or number)
% It is redundant info, and is listed her just for
% the sake of convenience.

% data=data(3:end,:)
T=data(:,2);
H=data(:,3);
m=length(T);
n=max(max(T), max(H));

netdata

DEGO
celldisp(A)
```

Output from Matlab:

```
% Matlab session
% Lecture 5, Sep 03, 2024

>> example1
DEGO =
   2    2    1    1    2
A{1} =
   1    2
A{2} =
   3    4
A{3} =
   5
A{4} =
   6
A{5} =
   7    8
DEGI =
   0    1    3    3    1
AI{1} =
   []
AI{2} =
   1
AI{3} =
   2    3    7
AI{4} =
   4    5    8
AI{5} =
   6
>>
```

We will implement several algorithms we discuss in class in Matlab.

Network Transformations

We discuss several ways to transform networks. The goal is to make the network amenable to algorithms requiring certain structure, while not changing the problem entirely.

Recall the model for min-cost flow:

$$\overset{b(i)}{(i)} \xrightarrow[x_{ij}]{c_{ij},\,u_{ij}} \overset{b(j)}{(j)}$$

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s.t. } \sum x_{ij} - \sum x_{ji} = b(i) \ \forall i$$

$$l_{ij} \le x_{ij} \le u_{ij} \ \forall (i,j)\in A.$$

1. Removing nonzero lower bounds $\to 0$

(We typically take $l_{ij} = 0$. But if $l_{ij} > 0$, we could transform the problem to an equivalent problem with $l_{ij} = 0$).

$$l_{ij} - l_{ij} \le x_{ij} - l_{ij} \le u_{ij} - l_{ij}$$

$$0 \le \underbrace{x_{ij}'}_{\text{new flow}} \le \underbrace{u_{ij}'}_{\text{new } u_{ij}}$$

$x_{ij} - l_{ij} = x_{ij}'$

so, $x_{ij} = x_{ij}' + l_{ij}$, and

$u_{ij}' = u_{ij} - l_{ij}$.

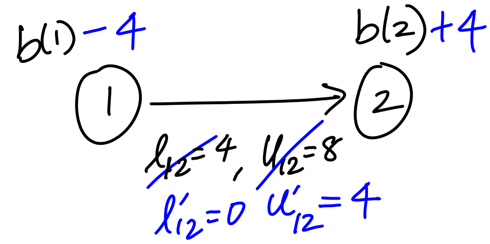
The flow balance constraint for node i becomes

$$\sum_{(i,j)} (x_{ij}' + l_{ij}) - \sum_{(j,i)} (x_{ji}' + l_{ji}) = b(i)$$

$$\Rightarrow \sum_{(i,j)} x_{ij}' - \sum_{(j,i)} x_{ji}' = \underbrace{b(i) - \sum_{(i,j)} l_{ij} + \sum_{(j,i)} l_{ji}}_{b'(i) \,\to\, \text{new } b(i) \text{ value}}$$

The $b(i)$ values are updated as follows. We **subtract** the l_{ij} values for all **outarcs** of node i, and **add** the l_{ji} values for all **inarcs** of node i.

Here is an illustration on a single arc: If we have to send at least 4 units along (1,2), we could imagine that much flow being "taken out of" $b(1)$, and being added to $b(2)$.



What about the objective function?

$$\sum_{(i,j)} c_{ij} x_{ij} = \sum_{(i,j)} c_{ij}(x'_{ij} + l_{ij}) = \sum_{(i,j)} c_{ij} x'_{ij} + \underbrace{\sum_{(i,j)} c_{ij} l_{ij}}_{\text{constant; does not depend on } x_{ij} \text{ values.}}$$

Adding a constant to the objective function does not change the optimal solution. Hence we will indeed find the optimal solution to the original problem.

Once you have the optimal solution x'_{ij}, we can recover the corresponding optimal solution x_{ij} by computing $x'_{ij} + l_{ij}$.