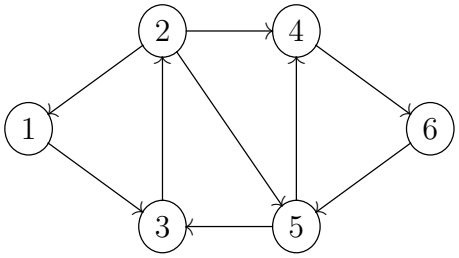
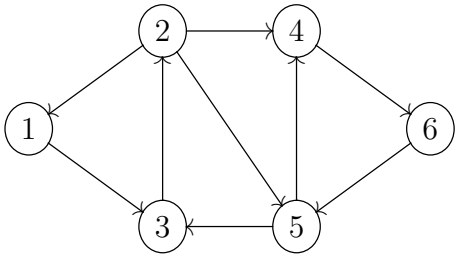
Network Optimization (Fall 2026): Midterm Exam

Name: _____

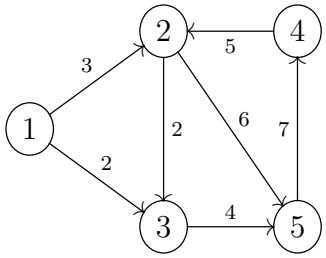
- There are **eight** problems on **four** pages.
- The total points (given in parentheses) add up to 105. You will be graded for 100 points.

1. (10) Give the BFS and DFS search trees for the following graph with starting node $s = 2$.

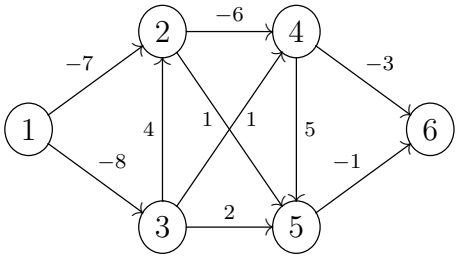


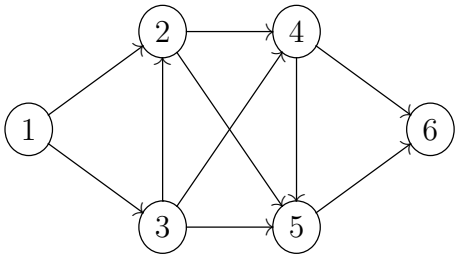


2. (10) Give a decomposition of the flow (the number on each arc (i, j) is the flow x_{ij}) into flows along paths and cycles. Is this decomposition unique?



3. (12) Find the *longest path* from node $s = 1$ to every other node in the graph (the number on arc (i, j) is c_{ij}). Show the optimal longest path distance labels and the longest path tree.





4. (14) Suppose you are given a min-cost flow problem on a network $G = (N, A)$ with $|N| = n,\ l_{ij} = 0\ \forall\,(i,j) \in A$, and $0 < u_{ij} < \infty\ \forall\,(i,j) \in A$. Further, the arcs going out of each supply node have enough capacity to carry all the supply from that node out, and the arcs coming in to each demand node have enough capacity to get flow in to that node to satisfy all its demand. In other words,

$$\sum_{(i,j)\in A} u_{ij} \geq b(i)\ \forall\ i \in N \text{ with } b(i) > 0, \quad \text{and} \quad \sum_{(j,i)\in A} u_{ji} \geq -b(i)\ \forall\ i \in N \text{ with } b(i) < 0.$$

How can you transform this min-cost flow problem to an equivalent *transportation problem*?

5. (14) Decide whether the following statement is TRUE or FALSE. Provide a brief justification.

   The *second* shortest s-t path in a network G has the *second* smallest total length among all directed s-t paths. We can find a second shortest s-t path in $O(nT)$ time, where $O(T)$ gives the running time to find the shortest s-t path in G.

6. (12) In each part, choose the best option. Give a **brief** justification.

   (a) Let $\ell_{\mathsf{DFS}}(i)$ and $\ell_{\mathsf{SP}}(i)$ denote the number of arcs in the path from node s to node i in the depth-first search (DFS) tree and the shortest path (SP) tree of a network $G = (N, A)$, respectively.

   i. $\ell_{\mathsf{DFS}}(i) \geq \ell_{\mathsf{SP}}(i) \;\; \forall\, i \in N$.
   ii. $\ell_{\mathsf{DFS}}(i) \leq \ell_{\mathsf{SP}}(i) \;\; \forall\, i \in N$.
   iii. $\ell_{\mathsf{DFS}}(i) \geq \ell_{\mathsf{SP}}(i)$ when $c_{ij} > 0 \;\forall\; (i,j) \in A$.
   iv. none of the above.

   (b) The decomposition of a flow in arcs to flows in paths and cycles

   i. can never be unique.
   ii. is unique if there are only path flows (and no cycle flows) in the decomposition.
   iii. is unique if there are only cycle flows (and no path flows) in the decomposition.
   iv. can be unique depending on the network in question.

7. (16) For each part, answer TRUE or FALSE. **Justify** your answer.

   (a) Let $p(i)$ denote the number of distinct paths from node s to node i in an acyclic graph G. We can find $p(i)$ for all nodes i in $O(m)$ time.

   (b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

8. (17) Consider a round-robin chess tournament with n players with each player playing every other player exactly once. A win scores 1 for the winner and 0 for the loser, while a draw scores $1/2$ for each player. We are given a set of final scores $(s_1, \ldots, s_n)$ for the players with $0 \leq s_i \leq n-1$. We want to check whether these scores are feasible (for example, in a three-player tournament, a set of final scores of $(2, 1/2, 2)$ is impossible). Model this problem as a network flow problem.