MATH 566: Lecture 16 (10/15/2026)

Offer to make-up midterm scores

- If you get 92% or more in Hws 6-11, then your midterm wt will be 5% (instead of 20%)
- If you get $\geq 85\%$ ($<92\%$) in Hws 6-11, then your midterm wt will be 10% (instead of 20%)

Today: * max flow application: matrix rounding
* residual network
* max flow algorithm

Read Section 6.3 in AMO for more modeling applications of the max flow problem.

We saw one application in the last lecture — the feasible flow problem. We discuss one more application today: the matrix rounding problem.

Matrix Rounding Problem

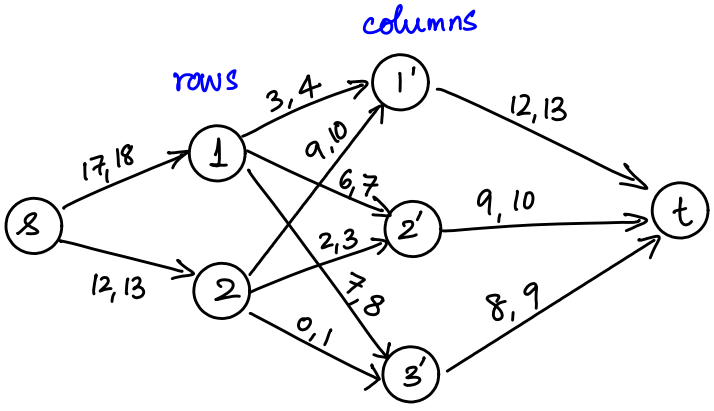
Given $m \times n$ matrix, $D = [d_{ij}]$ row sums α_i $(1 \le i \le m)$, and column sums β_j $(1 \le j \le n)$, where $d_{ij}, \alpha_i, \beta_j \in \mathbb{R}$, the goal is to round each element d_{ij} and row & column sums (α_i, β_j), such that the row sums of the rounded elements equal the rounded row sums, and similarly for column sums. For instance, consider the 2×3 matrix D:

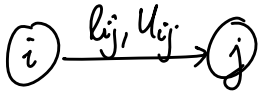
	1′	2′	3′	α_i
1	3.1	6.8	7.3	17.2
2	9.6	2.4	0.8	12.8
β_j	12.7	9.2	8.1	

Note that we can round each number up or down — e.g., 3.1 can be rounded to 3 or to 4.

$(\lfloor 3.1 \rfloor = 3, \lceil 3.1 \rceil = 4)$

We formulate this problem as a max flow problem as follows.





Here,
$N_1 = \{1, \ldots, m\}$: row nodes
$N_2 = \{1', 2', \ldots, n'\}$: column nodes

We add s, t, arcs (s,i), $i \in N_1$ (row nodes), (j,t) $\forall j \in N_2$ (column nodes), and set $\ell_{si} = \lfloor \alpha_i \rfloor$, $u_{si} = \lceil \alpha_i \rceil$, $\ell_{jt} = \lfloor \beta_j \rfloor$, $u_{jt} = \lceil \beta_j \rceil$. We also have (i,j), $1 \le i \le m$, $1 \le j \le n$, with $\ell_{ij} = \lfloor d_{ij} \rfloor$, $u_{ij} = \lceil d_{ij} \rceil$.

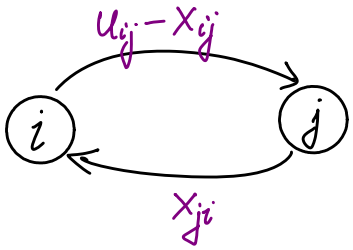
floor ↙ ↘ ceil

The x_{ij}'s in a max flow will be at ℓ_{ij} or u_{ij}, which are all integers. Further, flow balance at nodes in N_1 and N_2 impose the row and column sums, giving a consistent rounding.

Residual Network ("Remaining flow" network)

Algorithms for max flow will employ the concept of residual networks.







The diagram shows the components of the residual network corresponding to arc (i,j). Repeat for all arcs $(i,j) \in A$ to obtain $G(\bar{x})$, the residual network for the flow $\bar{x}$.

Notice that the residual network is defined for a particular flow $\bar{x}$.

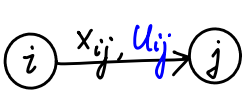
The **residual capacity**, r_{ij}, of arc (i,j) is given as

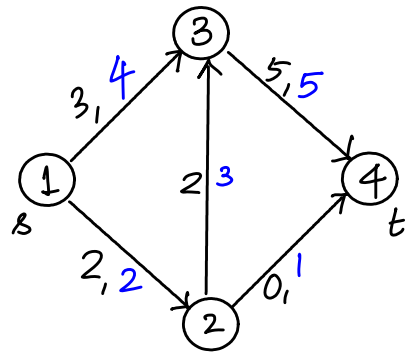
$$r_{ij} = u_{ij} - x_{ij} + x_{ji}.$$

This is the maximum additional flow we can send from i to j using arcs (i,j) and (j,i).

The residual network corresponding to flow $\bar{x}$ is $G(\bar{x})$, and it contains all (i,j) with $r_{ij} > 0$ on N, the same node set as the input graph G.

Example: Consider the network G with a flow (x_{ij} values) given

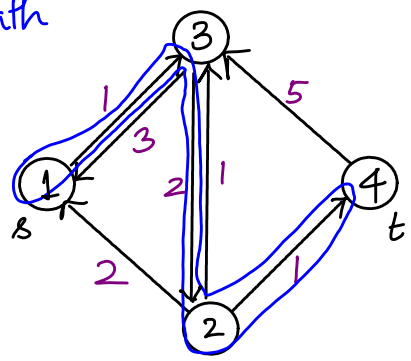




G with flow $\bar{x}$



s-t path



$G(\bar{x})$

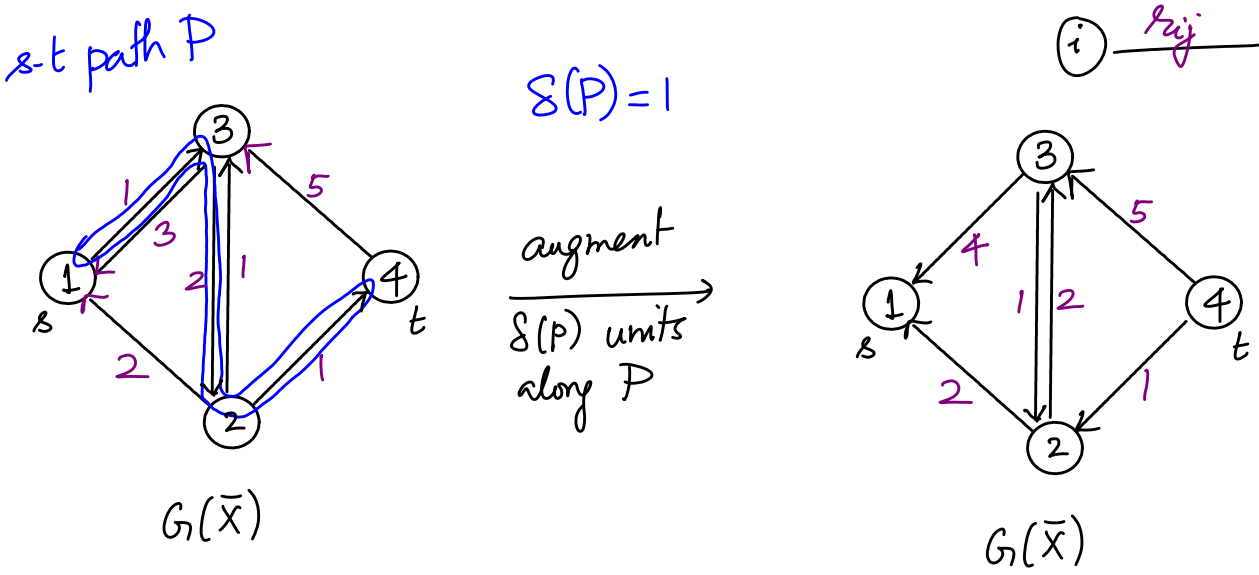
If there is a (directed) path from s to t in $G(\bar{x})$, we could push flow along this path. This idea is central to max flow algorithms — start with some flow $\bar{x}$, find $G(\bar{x})$, and send flow along s-t paths in $G(\bar{x})$. How much flow could we send from s to t along such an s-t path? We formalize these ideas into definitions now.

Def A directed path P from s to t in $G(\bar{x})$ is an **augmenting path**. The **residual capacity** of the augmenting path P is $\delta(P) = \min_{(i,j)\in P} \{r_{ij}\}$.

Note that $\delta(P) > 0$, as $(i,j) \in P \Rightarrow r_{ij} > 0$ by definition.

To **augment** along P is to send $\delta(P)$ units along each arc in P, and modify, $\bar{x}$, $G(\bar{x})$, i.e., r_{ij}'s, accordingly.

Let's consider the example again.



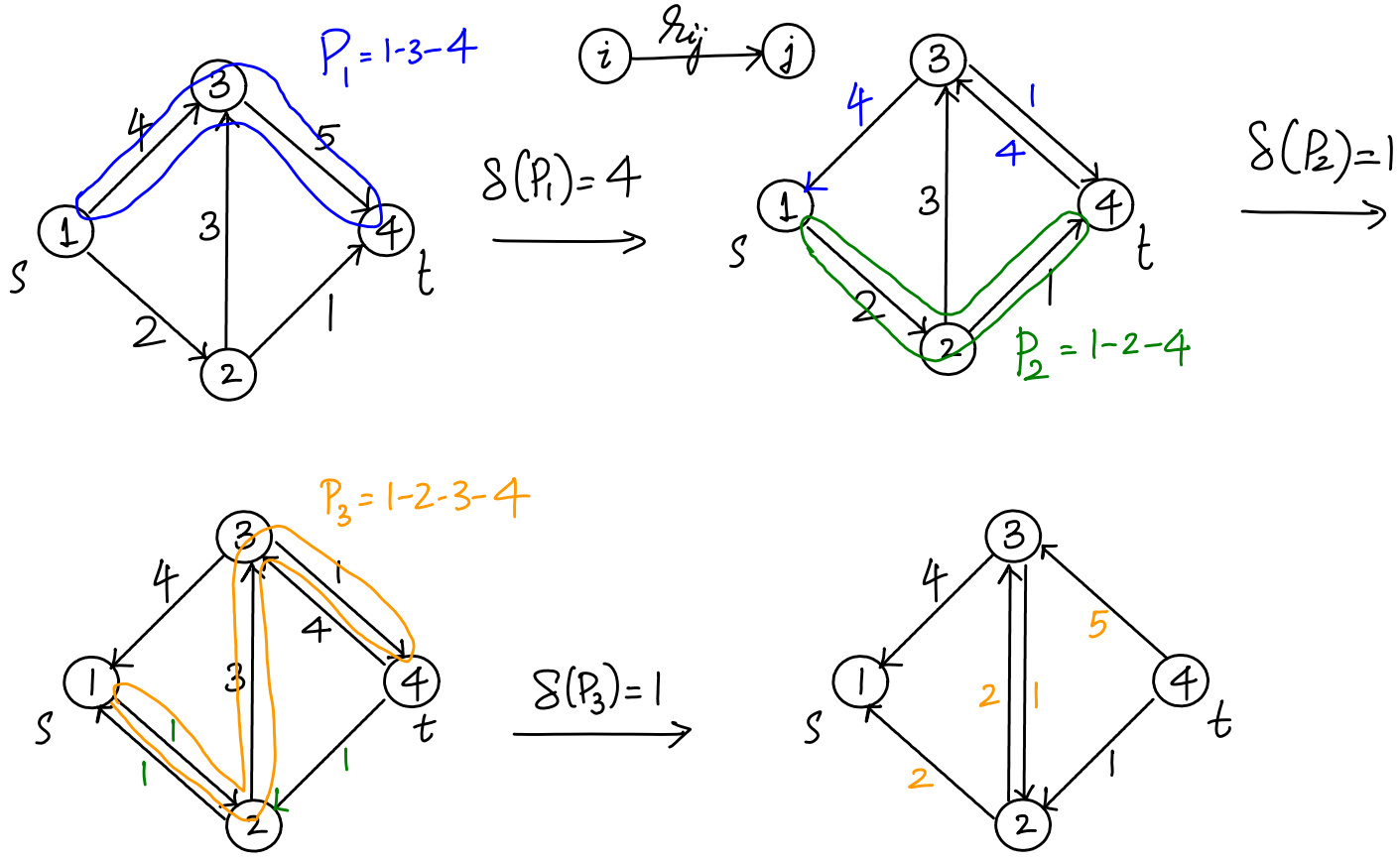
Now there are no more s-t paths in $G(\bar{x})$, which tell us that $\bar{x}$ is optimal!

The Generic Augmenting Path Algorithm (Ford-Fulkerson)

Assume $l_{ij} = 0 \ \forall (i,j) \in A$.

```
begin
    x̄ := 0̄;
    initialize G(x̄);
    while G(x̄) has a path P from s to t do
        augment δ(P) units of flow along P;
        update x̄, G(x̄);
    end_while
end_begin
```

Example We start with $\bar{x} = \bar{0}$, i.e., the zero flow. Notice that $G(\bar{x}) = G$ when $\bar{x} = \bar{0}$, assuming all $u_{ij} > 0$.



There are no more augmenting paths, and hence the flow is maximum. Notice that the value of the maximum flow is $\delta(P_1) + \delta(P_2) + \delta(P_3) = 4 + 1 + 1 = 6$.