Integer Optimization (Spring 2025): Homework 3

- You **must email your submission** as a **PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name** should identify you in the following manner. If you are Edward Murphy, you should name your submission EdwardMurphy_Hw3.pdf. If you want to add more bits to the title, e.g., Math567, you could name it EdwardMurphy_Math567_Hw3.pdf, for instance. But you should **start the file name with EdwardMurphy.** And it is **NOT "Edward Murphy" or "EddieMurphy" or ...**
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "EdwardMurphy Hw3 submission".**
- The total points (given in parentheses) add up to 130. You will be graded for 125 points.
- **This homework is due by 11:59 PM on Tuesday, February 25**.

1. (50) Represent the following sets. You should consider all the following three options.

   (*i*) Model without any extra variables.

   (*ii*) Model with extra continuous variables (here, sometimes you have to just use common sense, and not anything learned in systematic MIP modeling).

   (*iii*) Model with extra continuous and 0–1 variables.

   If it is not possible to represent a set as described in one of the options, prove that fact.

   (a) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \leq 1\}$.

   (b) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \geq 1\}$.

   (c) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \geq 1,\ -M \leq y_i \leq M\ \forall i\}$.

   (d) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n c_i|y_i| \leq 1\}$. Here the c_i are scalars, which may be positive or negative.

   (e) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n c_i|y_i| \leq 1,\ -M \leq y_i \leq M\ \forall i\}$. Here the c_i are scalars, which may be positive or negative.

   (f) $S = \{y \in \mathbb{Z}^n \mid Ay \leq b,\ y \neq y^*,\ y^* \in \mathbb{Z}^n \text{ is fixed}\}$.

2. (20) Consider the set S and the formulation P.

$$S = \{\mathbf{x} \in \{0,1\}^n \mid \text{at least } p \text{ of the } \mathbf{x}_i\text{'s are } 1\}, \quad \text{and}$$
$$P = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq \mathbf{x} \leq \mathbf{1},\ \mathbf{1}^T\mathbf{x} \geq p\},$$

   where $\mathbf{1}$ is the n-vector of ones. Assume p, n are positive integers such that $p \leq n$.

   Prove that P is a sharp (or ideal) formulation of S.

3. (60) Let

$$S = \{(x, y_1, y_2, y_3) \in \{0,1\}^4 \mid (x = 1) \Rightarrow \text{ at least two of the } y_i\text{'s are } 1\}.$$

Consider the following inequalities:

$$x \leq y_1 + y_2 \quad (1)$$
$$x \leq y_1 + y_3 \quad (2)$$
$$x \leq y_2 + y_3 \quad (3)$$
$$2x \leq y_1 + y_2 + y_3 \quad (4)$$
$$0 \leq x \leq 1 \quad (5)$$
$$0 \leq y_i \leq 1 \ (i = 1, 2, 3) \quad (6)$$

Now consider the following formulations:

- Formulation 1: bounds ((5),(6)) and (4);
- Formulation 2: bounds ((5),(6)), (4), and (1);
- Formulation 3: bounds ((5),(6)) (4), (1), and (2); and
- Formulation 4: all constraints listed (including bounds).

Prove the following statements.

(a) Formulations 1,2,3, and 4 are all valid formulations for S.

(b) Formulation 1 is not ideal for S.

(c) Formulation 2 is stronger than Formulation 1.

(d) Formulation 3 is stronger than Formulation 2.

(e) Formulation 4 is stronger than Formulation 3.

(f) Formulation 4 is ideal.