Integer Optimization (Spring 2025): Homework 5

- You **must email your submission** as a PDF file to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name** should identify you in this way. If you are Nomore Homeworky, you should name your file **NomoreHomeworky_Hw5.pdf**. If you want to add more bits to the title, e.g., Math567, you could name it NomoreHomeworky_Math567_Hw5.pdf, for instance. But you should **start the file name with NomoreHomeworky; NOT "Nomore Homeworky" or "Nom_Homeworky" or ...**
- Begin the **SUBJECT of your email** submission with the same `FirstnameLastname`, e.g., "Nomore-Homeworky Hw5 submission".
- **This homework is due by 11:59 PM on Tuesday, April 15.**

1. (25) For an integer $k \geq 1$, the ***k*-fold *I*-sum** of an $m \times n$ matrix A is the $(mk+n) \times nk$ matrix

$$\textcircled{\scriptsize I}_k A := \begin{bmatrix} I & I & \cdots & I \\ A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \qquad (I\text{-sum})$$

where I is the $n \times n$ identity matrix, k copies of which are included in the top row. Unspecified entries are zero. Does the I-sum preserve total unimodularity? If yes, give a proof. If not, give an counterexample (i.e., specify a matrix A that is totally unimodular (TU), but $\textcircled{\scriptsize I}_k A$ is not).

2. (25) Show that the node-edge incidence matrix of an *undirected* graph is TU if and only if the graph is bipartite. (There is one column for each edge (i, j), with a +1 in both rows i and j.)

3. (25) Let $A \in \mathbb{Z}^{m \times n}$, $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c} \in \mathbb{Z}^n$, and $d \in \mathbb{Z}$. Show that if the system $\{A\mathbf{x} \leq \mathbf{b},\ \mathbf{c}^T\mathbf{x} \leq d\}$ is totally dual integral (TDI), then the system $\{A\mathbf{x} \leq \mathbf{b},\ \mathbf{c}^T\mathbf{x} = d\}$ is also TDI.

4. Consider the **Jeroslow IP** (in its original form) for *odd* n.

$$\begin{array}{lrcl} \max\ z = & -x_{n+1} & & \\ \text{s.t.} & 2x_1 + \cdots + 2x_n + x_{n+1} & = & n \\ & x_j & \in & \{0,1\}, j = 1, \ldots, n+1. \end{array}$$

   (a) (15) Show that even if the optimal solution is known, i.e., the best lower bound z_ℓ is known, branch-and-bound still takes an exponential number of nodes to solve this problem. Will the analysis be different for the cases where the B&B branches first on x_{n+1} and when it branches first on one of the first n x_j variables?

   (b) (10) Now consider applying *branching on a constraint* to this IP. Identify a direction vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ such that branching on $\mathbf{a}^T\mathbf{x}$ will solve the IP in only two nodes.

   (c) We now explore whether cuts can help us solve the Jeroslow IP. For each method given below, indicate with justification if it will be helpful when applied to the Jeroslow IP. If yes, indicate how many times should it be applied before we solve the problem. If the method is not helpful, show why it is so.

      i. (8) CG cuts applied by themselves (i.e., without branch-and-bound).

      ii. (9) Lifted cover inequalities applied by themselves (i.e., without branch-and-bound).

      iii. (3) The Lovász-Schrijver (LS) procedure.