MATH 567: Lecture 11 (02/13/2025)

Today: * unimodularity and total unimodularity

Recall: integral polyhedra...

Theorem 7 (Hoffman 1974) A rational polytope P is integral iff for all integral vectors $\bar{w}$, the optimal value of $\max\{\bar{w}^T\bar{x} \mid \bar{x} \in P\}$ is an integer.

optimal $\bar{w}^T\bar{x}$

Proof ($\Rightarrow$) Let P be integral, i.e., all vertices are integral. Then $\max\{\bar{w}^T\bar{x} \mid \bar{x} \in P\}$ is integral for integral $\bar{w}$, as it occurs at a vertex.

($\Leftarrow$) Let $\max\{\bar{w}^T\bar{x} \mid \bar{x} \in P\}$ be integral for all integral $\bar{w}$.

Let $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a vertex of P,

and let $\bar{w}$ be an integral vector such that $\bar{v}$ is the unique optimal solution to $\max\{\bar{w}^T\bar{x} \mid \bar{x} \in P\}$.

We can assume $\bar{w}^T\bar{v} > \bar{w}^T\bar{u} + u_1 - v_1$ for all other vertices $\bar{u}$ of P. We can scale $\bar{w}$ by a large integer if needed (i.e., when $u_1 - v_1 < 0$).

Then we have $\bar{w}^T\bar{v} + v_1 > \bar{w}^T\bar{u} + u_1$ for all other vertices $\bar{u}$ of P. This inequality gives the following result.

$\Rightarrow$ $\bar{v}$ is the unique optimal solution for the objective function vector $\bar{w}' = [w_1+1, w_2, \ldots, w_n]^T = \bar{w} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, as

we get $\bar{w}'^T \bar{v} > \bar{w}'^T \bar{u}$ for all other vertices $\bar{u} \in P$.

By assumption, $\bar{w}^T \bar{v}$ and $\bar{w}'^T \bar{v}$ are integral.
Also, $\bar{w}, \bar{w}' \in \mathbb{Z}^n$. $\Rightarrow v_1 \in \mathbb{Z}$.

We repeat the argument for $v_2, v_3, \ldots, v_n$. $\square$

Can extend result to unbounded pointed polyhedra easily, and also to polyhedra in general (i.e., not pointed).

While this theorem specifies an if and only if condition for P to be integral, it does not appear easy to check. We will have to certify integrality of the optimal value for all $\bar{w} \in \mathbb{Z}^n$, for which the optimum exists. But could we specify some easier to check conditions which guarantee integrality?

Unimodularity and Total Unimodularity (TU)

We assume $A, \bar{b}$ are integral (in $A\bar{x}=\bar{b}$ or $A\bar{x} \le \bar{b}$).

Def Let $A \in \mathbb{Z}^{m \times n}$ with full row rank ($\text{rank}(A) = m \le n$).
A is **unimodular** if each basis of A has determinant ± 1.

$B_{m\times m}$ submatrix of A with $\text{rank}(B) = m$.

$A = [B\ N]$, $B \in \mathbb{Z}^{m \times m}$, $\text{rank}(B) = m$, $\det(B) = \pm 1 \Rightarrow B^{-1} \in \mathbb{Z}^{m \times m}$.

For $\bar{b} \in \mathbb{Z}^m$, $A\bar{x} = \bar{b}$ has all integer solutions.

Recall: basic feasible solutions (bfs's) for $\max\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x} \ge \bar{0}\}$ correspond to corner points (vertices) of $\{\bar{x} \mid A\bar{x} = \bar{b}, \bar{x} \ge \bar{0}\}$.

$A = [B\ N]$, $\bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}$. Set $\bar{x}_N = \bar{0}$, solve for $\bar{x}_B$.

$$A\bar{x} = \bar{b} \Rightarrow B\bar{x}_B + N\underbrace{\bar{x}_N}_{=\bar{0}} = \bar{b}.$$

We get $\bar{x}_B = B^{-1}\bar{b}$

$\bar{x} = \begin{bmatrix} B^{-1}\bar{b} \\ \bar{0} \end{bmatrix}$ is a basic solution.

If a basic solution $\bar{x}$ satisfies $\bar{x} \ge \bar{0}$, then it's feasible, and hence is a bfs. Each bfs corresponds to a vertex (or corner point) of P.

We can use the idea of bfs to give a correspondence between unimodularity of A and integrality of P.

Theorem 8 $A \in \mathbb{Z}^{m \times n}$, $\operatorname{rank}(A) = m$. Then $P = \{\bar{x} \mid A\bar{x} = \bar{b}, \underbrace{\bar{x} \geq \bar{0}}_{\text{ensures } P \text{ is pointed}}\}$ is integral for all $\bar{b} \in \mathbb{Z}^m$ iff A is unimodular.

$$A = [B \; N], \quad \bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}, \quad \text{set } \bar{x}_N = 0 \Rightarrow \bar{x}_B = B^{-1}\bar{b} \in \mathbb{Z}^m.$$

If A is not unimodular, $\det(B) \neq \pm 1$ for some basis B of A, and hence the corresponding basic solution will not be integral for all $\bar{b} \in \mathbb{Z}^m$.

Note that $B^{-1}\bar{b}$ might be integral for some $\bar{b} \in \mathbb{Z}^m$ in this case, e.g., when $|\det(B)| = 2$, and $\bar{b} \in 2\mathbb{Z}^m$. But the result will not hold for all $\bar{b} \in \mathbb{Z}^m$.

We now define a stronger (tighter) property of A, which guarantees integrality for polyhedra defined in more general forms. In particular, we would like to relax the requirement that $\operatorname{rank}(A) = m$ (i.e., be able to consider more general "shapes" of A, e.g., more tall than wide, i.e., $m > n$).

Def A matrix A is **totally unimodular** (TU) if every square submatrix of A has determinant $-1, 0,$ or 1. In particular, $A_{ij} \in \{-1, 0, 1\}\ \forall\, i, j$.

e.g., $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ is unimodular, but not TU.

$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is TU (and unimodular here).

$A = \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \\ \begin{matrix} 1 & 1 & 0 \end{matrix} & 1 \end{bmatrix}$ is not TU, as $\det(A_{1:3,1:3}) = -2$.

Notice that we have to check **all** square submatrices in the worst case. But it turns out we could check whether a matrix is TU or not in polynomial time by Seymour's decomposition algorithm, which runs in $O(n^3)$ time. But no implementation is known. Another algorithm by Truemper runs in $O(n^5)$ time, but is implemented, though.

→ check out https://discopt.github.io/cmr/ [Combinatorial Matrix Recognition (CMR) library]

Here is a result that connects total unimodularity and integrality of polyhedra.

Theorem 9 [Hoffman & Kruskal, 1956]: Let $A \in \mathbb{Z}^{m \times n}$. Then $P = \{\bar{x} \mid A\bar{x} \le \bar{b}, \bar{x} \ge \bar{0}\}$ is integral $\forall \bar{b} \in \mathbb{Z}^m$ for which $P \ne \emptyset$ iff A is totally unimodular (TU).

Proof We need a proposition first.

Proposition The following statements are equivalent.

(i) A is TU.

(ii) A^T is TU. → determinant is preserved under transposes

(iii) $[A\ I_m]$ is unimodular ↪ full row rank now!

(iv) $\begin{bmatrix} A \\ -A \\ I_n \\ -I_n \end{bmatrix}$ is TU.

Back to proof of Theorem 9.

$P = \{\bar{x} \mid A\bar{x} \le \bar{b}, \bar{x} \ge \bar{0}\}$ is integral $\iff$

$\mathrm{Proj}_{\bar{x}}(P^z)$ is integral, where $P_z = \{\bar{z} \mid [A\ I]\bar{z} = \bar{b}, \bar{z} \ge \bar{0}\}$

is integral, where $\bar{z} = \begin{bmatrix} \bar{x} \\ \bar{s} \end{bmatrix}$. → slack variables

Now apply Theorem 8, which gives that P_z is integral iff $[A\ I]$ is unimodular, which is true iff A is TU, by the proposition above. $\square$

Here is another characterization of TU and integral polyhedra.

Theorem 10 Let $A \in \mathbb{Z}^{m \times n}$. A is TU iff $P = \{\bar{x} \mid A\bar{x} \leq \bar{b}\}$ is integral $\forall\, \bar{b} \in \mathbb{Z}^m$ for which $P \neq \phi$.

$\bar{x}$ is urs (unrestricted in sign) here. We can replace $\bar{x}$ by $\bar{x}^+ - \bar{x}^-$, where $\bar{x}^+, \bar{x}^- \geq \bar{0}$, and use Theorem 9.

The upshot is that as long as the constraint matrix of the LP is TU, we're in good shape. The form of the LP – general or standard – does not matter.

Operations that preserve total unimodularity

1. Swap two rows (or columns).
2. Taking transpose.
3. Scaling a row/column by −1.
4. Pivoting, i.e., converting a column to a unit vector using EROs.
5. Adding a zero row/column, or a singleton row/column with the single nonzero entry being ±1.
6. Repeating a row/column.

⋮