MATH 567: Lecture 12 (02/18/2025)

Today: * sufficient conditions for TU
* min-cost flow
* LP duality, TDI

Some details on the operations that preserve TU...

Example of pivoting:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix}$$

det $= -2$

this was the example for a non-TU matrix introduced in Lecture 11...

Equivalently, replacement EROs of the form

$R_i \leftarrow R_i \pm R_j$ preserve TU.

By "preserve", we mean that (when $\tilde{A}$ is obtained by performing the operation on A)

A is TU $\Leftrightarrow$ $\tilde{A}$ is TU, and

A is not TU $\Leftrightarrow$ $\tilde{A}$ is not TU.

Seymour's decomposition theorem uses these and a few other operations that preserve TU (k-sum, for $k=1,2,3$). It decomposes the task of checking for TU of A into doing the same for several small submatrices obtained by these TU-preserving operations. The TU of these small matrices can be checked immediately (in constant time).

Sufficient Conditions for TU

Theorem 11 Let $A \in \{-1,0,1\}^{m \times n}$, with each column having at most one +1 and one −1. Then A is TU.

Proof We prove the result using induction on k for $k \times k$ submatrix B of A.

$k=1$. $A_{ij} \in \{-1,0,1\}$. ✓

Induction for $k \geq 2$ (going from k to $k+1$)

If B has a row/column of all zeros, $\det(B)=0$. ✓

If B has a row/column with one nonzero (± 1), we can expand along that row/column, and using the induction assumption, we get $\det(B) \in \{-1,0,1\}$.

If every column of B has exactly two nonzeros, then adding all rows of B gives the zero vector. Hence $\det(B)=0$.

as every column has one +1 and −1.

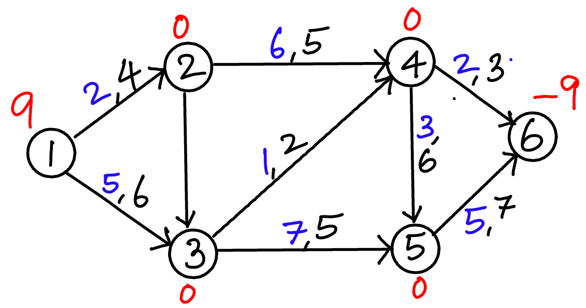
$B\bar{1} = \bar{0}$, where $\bar{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is a non-trivial solution to $B\bar{x}=\bar{0} \Rightarrow \det(B)=0$.

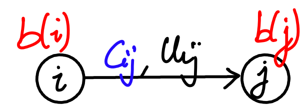
□

Min-Cost Flow (MCF) on a directed Network

Network matrices satisfy the above sufficient conditions. The node-arc incidence matrix of a directed network (or graph) in the context of the min-cost flow problem is an example.

$G = (V, E)$





c_{ij} : unit cost on (i,j)

u_{ij} : capacity (upperbound) of flow on (i,j)

Assume: total supply = total demand.

Each node i has supply/demand $b(i)$. If $b(i) > 0$, i is a supply node, and if $b(i) < 0$, i is a demand node. If $b(i) = 0$ then i is a transshipment. The goal is to satisfy demand using the supply by transporting the good through the arcs at the least total cost while honoring arc capacities.

Here is the LP: x_{ij} = flow in arc (i,j).

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \underbrace{\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji}}_{\text{outflow} - \text{inflow}} = \underbrace{b(i)}_{\text{supply/demand}} \quad \forall i \in V$$

$$0 \le x_{ij} \le u_{ij} \quad \forall (i,j) \in E.$$

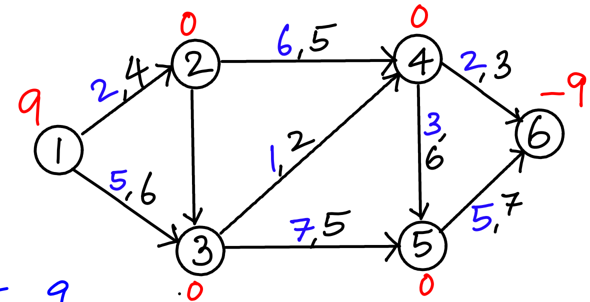
we assume lower bounds are all zero

With $\bar{x} = \begin{bmatrix} \vdots \\ x_{ij} \\ \vdots \end{bmatrix}$, the LP can be written as

$$\begin{aligned} \min \quad & \bar{c}^T \bar{x} \\ \text{s.t.} \quad & A\bar{x} = \bar{b} \\ & I\bar{x} \leq \bar{u} \\ & \bar{x} \geqslant \bar{0} \end{aligned} \qquad \bar{b} = \begin{bmatrix} \vdots \\ b(i) \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} A \\ I \end{bmatrix} \bar{x} \begin{pmatrix} = \\ \leq \end{pmatrix} \begin{bmatrix} \bar{b} \\ \bar{u} \end{bmatrix}$$

where A is the node-arc incidence matrix of G, which is guaranteed to be TU as it satisfies the sufficient condition for TU in Theorem 11.



$$A = \begin{array}{c|c} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 1 & & & & & & & \\ -1 & & 1 & 1 & & & & & \\ & -1 & -1 & & 1 & 1 & & & \\ & & & -1 & -1 & & 1 & 1 & \\ & & & & & -1 & -1 & & 1 \\ & & & & & & & -1 & -1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & (i,j) \\ \begin{array}{c} i \\ \\ j \end{array} & \begin{bmatrix} +1 \\ \\ -1 \end{bmatrix} \end{array}$$

A is TU. From A we get the constraint matrix $\begin{bmatrix} A \\ I \end{bmatrix}$ by adding singleton rows (one for each $(i,j) \in E$), which all preserve TU.

Hence, if $\bar{b}$ and $\bar{u}$ are integral, then the min-cost-flow problem is guaranteed to have integer optimal solutions.

This result does not necessarily hold for undirected graphs.

The same result holds for other problems on directed networks, e.g., shortest path, max flow, transportation, etc. problems. But there are efficient algorithms for each problem — which are faster than solving them as LPs.

We now introduce the concept of total dual integrality, which is a more general concept than TU.

We first do a quick review of LP duality.

Review of LP Duality

For every linear program (LP), there is another associated LP called its dual LP. Solving the original (primal) LP is equivalent to solving its dual LP, and there are many results relating the two LPs and their interplay.

Here is an example:

(P) primal LP

$$
\begin{aligned}
\max\ & z = 2x_1 + 3x_2 \\
\text{s.t.}\ & x_1 + 4x_2 \le 5 \quad && y_1 \ge 0 \\
& -2x_1 + 3x_2 = 4 && y_2 \text{ urs} \\
& 5x_2 \ge 7 && y_3 \le 0 \\
& x_1 \le 0,\ x_2 \ge 0 \\
& \le \qquad \ge
\end{aligned}
$$

(D) dual LP

$$
\begin{aligned}
\min\ & w = 5y_1 + 4y_2 + 7y_3 \\
\text{s.t.}\ & y_1 - 2y_2 \le 2 \\
& 4y_1 + 3y_2 + 5y_3 \ge 3 \\
& y_1 \ge 0,\ y_2 \text{ urs},\ y_3 \le 0
\end{aligned}
$$

Normal vars and normal constraints

max-LP: $\le$ is normal

"maximize revenue s.t. upper bound on raw materials."

min-LP: $\ge$ is normal

"minimize cost s.t. meeting demand, i.e., produce at least a lower bound # units."

≥ 0 vars are always normal.

(opposite to) normal vars correspond to (opposite to) normal constraints in the dual LP. And urs vars correspond to $=$ constraints.

Table of Primal-Dual relationships

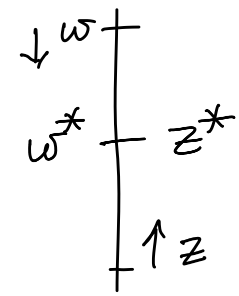
	Primal $\longleftrightarrow$	Dual	
	min	max	
variables	≥ 0	$\leq$	$\rightarrow$ normal
	≤ 0	$\geq$	$\rightarrow$ opposite to normal
	urs	$=$	
constraints	$\geq$	≥ 0	
	$\leq$	≤ 0	
	$=$	urs	

LP duality in matrix form:

$$(P) \quad \begin{array}{ll} \max & z = \bar{c}^T \bar{x} \\ \text{s.t.} & A\bar{x} \leq \bar{b} \quad \bar{y} \geq \bar{0} \end{array}$$

$$\begin{array}{ll} \min & w = \bar{b}^T \bar{y} \\ \text{s.t.} & A^T \bar{y} = \bar{c} \\ & \bar{y} \geq \bar{0} \end{array} \quad (D)$$

One could imagine pushing z up, and pulling w down. Every value of z (i.e., for each feasible solution) lies below every value of w. When they are equal, we have optimality for both primal and dual LPs.



Results on LP duality

1. Weak duality: $z = \bar{c}^T \bar{x} \leq \bar{b}^T \bar{y} = w$ for any feasible $\bar{x}, \bar{y}$ for (P) and (D), respectively.

2. Strong duality: $\bar{c}^T \bar{x} = \bar{b}^T \bar{y} \iff \bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively.

The default simplex method tries to push z up to optimality by working on (P). There is an equivalent dual simplex method that pushes w down by working on (D). There are also primal-dual methods which work on both ends, trying to push both z and w to optimality simultaneously.

Total Dual Integrality (TDI)

LP duality: $\max\{\bar{c}^T\bar{x} \mid A\bar{x} \le \bar{b}\} = \min\{\bar{b}^T\bar{y} \mid A^T\bar{y} = \bar{c},\ \bar{y} \ge \bar{0}\}$ —— $(*)$

Def A system $A\bar{x} \le \bar{b}$ is **totally dual integral** (TDI) if the minimum in $(*)$ is achieved by an integral $\bar{y}$ for each integral $\bar{c}$ for which the optimum exists.