MATH 567: Lecture 13 (02/20/2025)

Today: * Total Dual Integrality (TDI)
* AMPL

Total Dual Integrality (TDI) (Recall...)

LP duality: $\max\{\bar{c}^T\bar{x} \mid A\bar{x} \le \bar{b}\} = \min\{\bar{b}^T\bar{y} \mid A^T\bar{y} = \bar{c}, \bar{y} \ge \bar{0}\}$ —— (*)

Def A system $A\bar{x} \le \bar{b}$ is **totally dual integral** (TDI) if the minimum in (*) is achieved by an integral $\bar{y}$ for each integral $\bar{c}$ for which the optimum exists.

We present the first result connecting TDI systems and integral polyhedra — its implication goes only one way, i.e., it is not an "if-and-only-if" result.

Theorem 12 [Hoffman, 1974] Let $A\bar{x} \le \bar{b}$ be a TDI system such that $P = \{\bar{x} \mid A\bar{x} \le \bar{b}\}$ is a rational polytope and $\bar{b}$ is integral. Then P is an integral polytope.

Proof As $\bar{b}$ is integral, and $A\bar{x} \le \bar{b}$ is TDI, $\max\{\bar{c}^T\bar{x} \mid A\bar{x} \le \bar{b}\}$ is integral for all integral $\bar{c}$. Then use Theorem 7.

Note that: TDI is the property of a specific system of inequalities used to describe a polyhedron, and not of the polyhedron itself. So, the same polyhedron could be described by both a TDI system and another system which is not TDI!

Example 1

$$\begin{array}{llll} & \max & c_1x_1 + c_2x_2 & \\ (P) & \text{s.t.} & x_1 + x_2 \le b_1 & y_1 \ge 0 \\ & & \quad\;\; x_2 \le b_2 & y_2 \ge 0 \\ & & \;\;= \qquad = & \end{array}$$

$$\begin{array}{lll} \min & b_1y_1 + b_2y_2 & \\ \text{s.t.} & y_1 & = c_1 \\ (D) & y_1 + y_2 & = c_2 \\ & y_1, y_2 \ge 0 & \end{array}$$

Let $b_1, b_2 \in \mathbb{Z}_{\ge 0}$. For $c_1, c_2 \in \mathbb{Z}$, we solve system in (D) to get

$$\left.\begin{array}{l} y_1 = c_1 \quad \in \mathbb{Z} \\ y_2 = c_2 - c_1 \in \mathbb{Z} \end{array}\right\} \Rightarrow \text{Solution to (D) is integral, } \underline{\text{when it exists,}}$$

i.e., when $c_1 \ge 0$, $c_2 \ge c_1$.

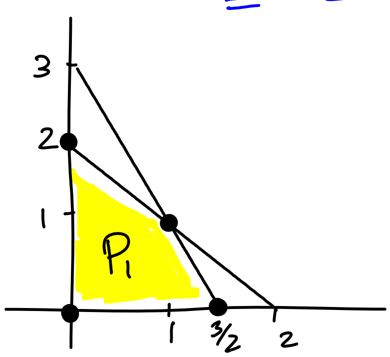
else (D) is infeasible

$\Rightarrow$ The system $\left\{\begin{array}{l} x_1 + x_2 \le b_1 \\ \quad\;\; x_2 \le b_2 \end{array}\right\}$ is TDI.

Example 2

$$\begin{array}{llll} & \max & z = c_1x_1 + c_2x_2 & \\ & \text{s.t.} & x_1 + x_2 \le 2 & y_1 \ge 0 \\ & & 2x_1 + x_2 \le 3 & y_2 \ge 0 \\ (P_1) & & -x_1 \qquad \le 0 & y_3 \ge 0 \\ & & \qquad -x_2 \le 0 & y_4 \ge 0. \\ & & \;\;= \qquad = & \end{array}$$

$$\begin{array}{lll} \min & w = 2y_1 + 3y_2 & \\ \text{s.t.} & y_1 + 2y_2 - y_3 & = c_1 \\ (D) & y_1 + y_2 \qquad - y_4 & = c_2 \\ & y_i \ge 0 \;\forall i & \end{array}$$



$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & c_1 \\ 1 & 1 & 0 & -1 & c_2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & c_1 \\ 0 & -1 & 1 & -1 & c_2 - c_1 \end{array}\right]$$

$$\xrightarrow[\text{then } -R_2]{R_1 + 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 2c_2 - c_1 \\ 0 & 1 & -1 & 1 & c_1 - c_2 \end{array}\right] \text{ gives}$$

$$\left.\begin{array}{l} y_1 = 2c_2 - c_1 - y_3 + 2y_4 \\ y_2 = c_1 - c_2 + y_3 - y_4 \end{array}\right\} \text{does not help much!}$$

But, for $c_1 = 1, c_2 = 0$, (D) has a unique <u>optimal solution @ $y_2 = y_4 = \frac{1}{2}$, $w^* = \frac{3}{2}$.</u>

Note that $\max\{x_1 \mid \bar{x} \in P_1\} = 3/2$

P_1 is not integral!

So, the system describing (P_1) is not TDI!

We get this result also as a contrapositive result to Theorem 12.

It may not be surprising that the polyhedron (P_1) is non-integral and the system describing (P_1) is not TDI. But we could have the reverse case as well — the polyhedron is integral but the system is still not TDI!

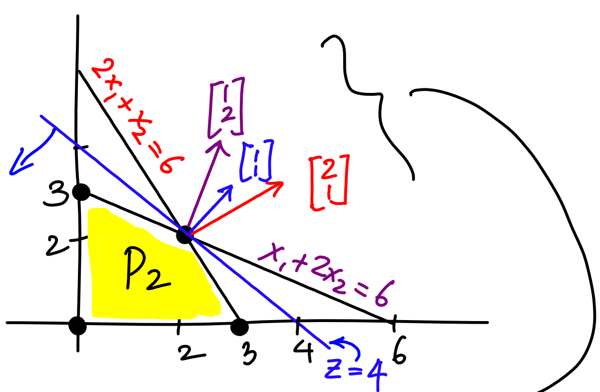
Example 3

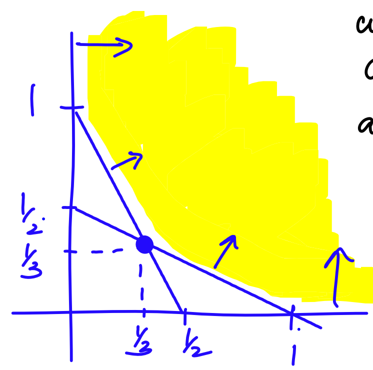
$$\max\ z = x_1 + x_2$$
$$(P_2)\quad \text{s.t.} \begin{cases} x_1 + 2x_2 \le 6 \\ 2x_1 + x_2 \le 6 \\ x_1, x_2 \ge 0 \end{cases} \begin{matrix} y_1 \ge 0 \\ y_2 \ge 0 \end{matrix}$$
$$\quad\quad\quad\quad\quad\ \ \ge \quad \ge$$

$z^* = 4$ at $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\min\ w = 6y_1 + 6y_2$$
$$\text{s.t.}\quad \begin{aligned} y_1 + 2y_2 &\ge 1 \\ 2y_1 + y_2 &\ge 1 \\ y_1, y_2 &\ge 0 \end{aligned} \quad (D)$$

$w^* = 4$ at $\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$





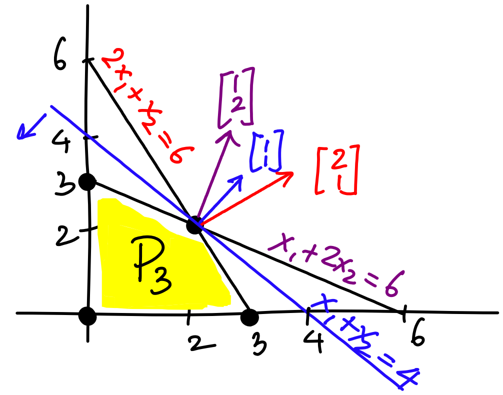
We could treat $x_i \ge 0$ as regular inequalities, use y_3, y_4 for them, and still get $y_1 = y_2 = \frac{1}{3}$ as the unique optimal solution!

Normal vectors of the constraints and $z = x_1 + x_2$.

We should be able to express $\begin{bmatrix}1\\1\end{bmatrix}$ as an integer linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$, for an integer (optimal) solution to exist. Hence (P_2) is not TDI.

So, (P_2) is not TDI, even though polytope is integral.

But we can describe (P_2) (the polytope) by another system of inequalities (P_3), which is indeed TDI.



$$\max\ z = x_1 + x_2$$

$$\text{s.t.}\quad (P_3)\begin{cases} x_1+2x_2 \le 6 \\ 2x_1 + x_2 \le 6 \\ x_1 + x_2 \le 4 & y_3 \\ x_1, x_2 \ge 0 \end{cases}$$

$y_3 = 1,\ y_1 = y_2 = 0$ is an integral optimal solution to (D), showing is TDI.

Now, $\begin{bmatrix}1\\1\end{bmatrix}$ can indeed be expressed as an integer linear combination of $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix}$.

The power of TDI lies oftentimes more on the mathematical side than on the computational/practical side. Knowing that a polyhedron can be described by a TDI system could be useful in proving certain related results.

Theorem 8.13 (Bertsimas-Weismantel)

Every rational polyhedron P can be described as a TDI system of the form $A\bar{x} \le \bar{b}$ with A integral.

Corollary A rational polyhedron P is integral iff there exists a TDI system describing P of the form $A\bar{x} \le \bar{b}$ with $A, \bar{b}$ integral.

AMPL

See AMPL handout posted on the course web page.

For the Farmer Jones LP (used as the first example), one could use n for the # crops in place of a set of crops. See the course web page for AMPL files using # crops.

Integer programming example

Knapsack feasibility problem: $\beta' \leq \bar{a}^T \bar{x} \leq \beta$
$\bar{x} \in \{0,1\}^n$

$a_1 x_1 + \cdots + a_n x_n$

goal is to check feasibility: $\overset{?}{\exists}\ \bar{x} \in \{0,1\}^n$ satisfying the knapsack inequalities?

There is no objective function (or, one could use a dummy objective function). The goal is to find an integer feasible point $\bar{x}$ satisfying the knapsack bounds, or prove there are no integer feasible solutions. Indeed, the latter case represents the worst case instances for most IP algorithms.

For the instance illustrated in class, we had $n=50$, and all the numbers (50 a_i's, β', and β) were available in a text file. The data could be read into ampl using the **read** command.