MATH 567: Lecture 14 (02/25/2025)

Today: * Branch-and-Bound (B&B)

Branch-and-Bound (B&B)

We describe a generic branch-and-bound algorithm for the problem of finding $z(S) = \max\{\bar{c}^T\bar{x} \mid \bar{x} \in S\}$. S here need not be a polyhedron, or disjoint collections of polyhedra, or even polyhedra intersected with $\mathbb{Z}^n$. It could be quite general.

We assume that

* we can divide a subproblem $T \subseteq S$, and
* we can compute lower and upper bounds

$$z_\ell(T) \le z(T) \le z_u(T)$$

for $z(T) = \max\{\bar{c}^T\bar{x} \mid \bar{x} \in T\}$.

We should be able to compute these bounds easily, i.e., in polynomial time. For MIP/IP, we usually solve LP relaxations, i.e., the problems without integrality restrictions.

We will describe how to maintain and update these bounds so as to arrive at the final answer.

Here is the generic algorithm (for $z = \max\{\bar{c}^T\bar{x} \mid \bar{x} \in S\}$)

Step 0 Let $\mathcal{L} = \{S\}$. (the list of (sub) problems).
compute $z_\ell(S), z_u(S)$.

Step r (i) Remove a subproblem $T \in \mathcal{L}$. $(\mathcal{L} \leftarrow \mathcal{L}/\{T\})$

(ii) Divide T as $T = T_1 \cup \cdots \cup T_k$;
compute $z_\ell(T_i)$, $z_u(T_i)$, $i = 1, \ldots, k$.
set $\mathcal{L} = \mathcal{L} \cup \{T_1, \ldots, T_k\}$.

(iii) Let $z_\ell(S) = \max\{z_\ell(S), \max\{z_\ell(T) \mid T \in \mathcal{L}\}\}$

save the solution $\bar{x}$ here

integer feasible

(iv) Prune all $T \in \mathcal{L}$ with

throw away; i.e., remove from $\mathcal{L}$

$z_u(T) \leq z_\ell(S)$;

(v) if $\mathcal{L} = \phi$ STOP;
else
set $z_u(S) = \max\{z_u(T) \mid T \in \mathcal{L}\}$;
end

Correctness of the generic B&B algorithm

Claim 1 At any time

$$\{\bar{x} \in S' \mid \bar{c}^T\bar{x} > z_\ell(S)\} \subseteq \mathcal{L}.$$

* true in the beginning
* maintained in Step (iv), where we prune a problem from $\mathcal{L}$.

remove

Claim 2 Update in Step (v) is correct.

Case 1: $z(S) > z_\ell(S)$

If $\bar{x}^*$ has $\bar{c}^T\bar{x}^* = z(S)$, then by Claim 1, $\bar{x}^* \in T$ for $T \in \mathcal{L}$. Then $z_u(T) \geq \bar{c}^T\bar{x}^* = z(S)$.

Case 2: $z(S) = z_\ell(S)$.

Since we are already past Step (iv), we must have

$$z_u(T) > z_\ell(S) = z(S) \quad \forall\, T \in \mathcal{L}.$$

□

Let's compare the updates of z_ℓ, z_u:

consider current best value, and those from all children

$$z_\ell(S) = \max\{z_\ell(S), \max\{z_\ell(T) \mid T \in \mathcal{L}\}\}.$$

$$z_u(S) = \max\{z_u(T) \mid T \in \mathcal{L}\}.$$ consider values only from the children

More specific details for $k=2$ (we create two subproblems: $S = S_1 \cup S_2$)

1. Partition S: let $x_i \in \mathbb{Z}$ in S, but $x_i = k + \delta$ in the LP solution for S, where $k \in \mathbb{Z}$, $0 < \delta < 1$. Then we can set $S_1 = S \cap \{\bar{x} \mid x_i \leq k\}$ and $S_2 = S \cap \{\bar{x} \mid x_i \geq k+1\}$.

2. Find z_{uj}, $j=1,2$ by solving the LP relaxations of $\max \{\bar{c}^T\bar{x} \mid \bar{x} \in S_j\}_{j=1,2}$. If infeasible, set $z_{uj} = -\infty$.

3. Find $z_{\ell j}$, $j=1,2$. Try to find any integer feasible solution $\bar{x}^j \in S_j$, $j=1,2$. If we cannot find an integer solution $\bar{x}^j$, set $z_{\ell j} = -\infty$.

We will have

$$\max_j \{z_j\} = z \qquad \text{—— (1)}$$

$$\max_j \{z_{\ell j}\} \leq z \leq \max_j \{z_{uj}\} \qquad \text{—— (2)}$$

We keep updating the lower and upper bounds by taking the max in each case.

In general, $z_{\ell j}$ comes from an integer feasible solution, and z_{uj} comes from a **relaxation**. The typical relaxation involves relaxing, i.e., ignoring the integrality constraints. But one could ignore any subset of constraints.

Another example of a relaxation:

$S = \{\bar{x} \mid \bar{x}$ is the incidence vector of a TSP tour$\}$.

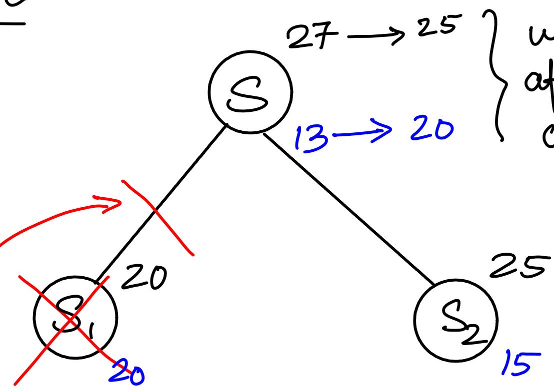
To get a relaxation of S, throw away the subtour constraints.

How do we use (2): update of bounds

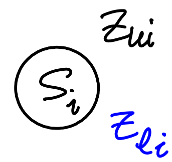
Three examples

1.

Notation for pruning: X or



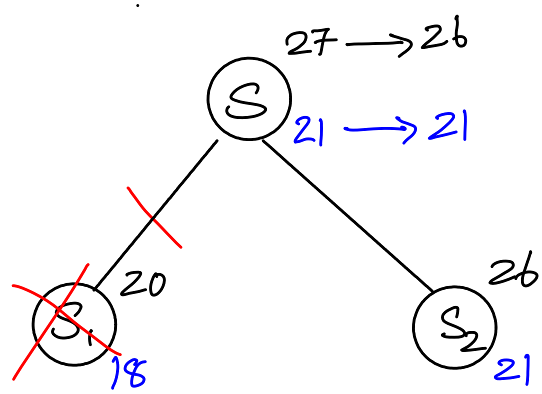
update bounds after handling child nodes



prune S_1 by **optimality**.

We have equality of the bounds in S_1. We cannot improve the solution any more, so can prune it. Also, there is no need to branch any more.

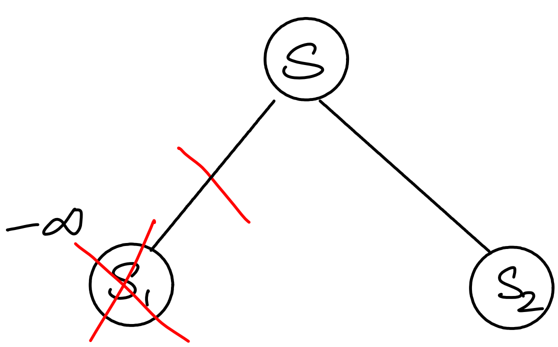
2.



prune S_1 by **bound**.

$z_{u1} = 20 < z_l = 21$. We cannot possibly find a better solution by branching on S_1 any more. So we prune it.

3.



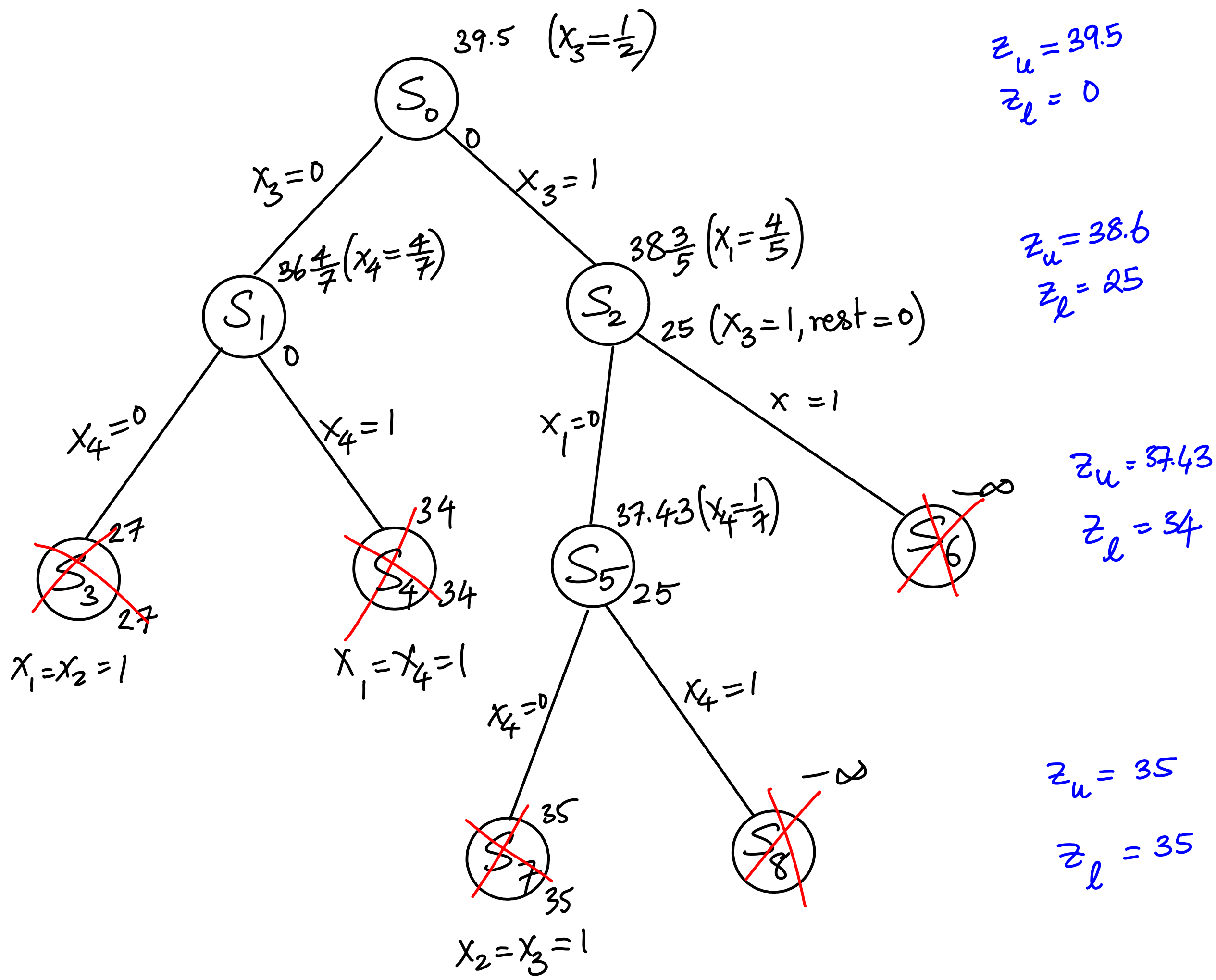
prune S_1 by **infeasibility**.

S_1 is not feasible even as an LP – no point considering it any further.

Illustration of B&B on $n=4$ knapsack problem

Prob 3, Chap 7 of Wolsey, Integer programming

$$\max\ z = 17x_1 + 10x_2 + 25x_3 + 17x_4$$
$$\text{s.t.}\quad 5x_1 + 3x_2 + 8x_3 + 7x_4 \le 12$$
$$\bar{x} \in \{0,1\}^4$$



S_3, S_4, S_7 are pruned by optimality, while S_6 and S_8 are pruned by infeasibility. No nodes are pruned by bound here. See the AMPL session for details.