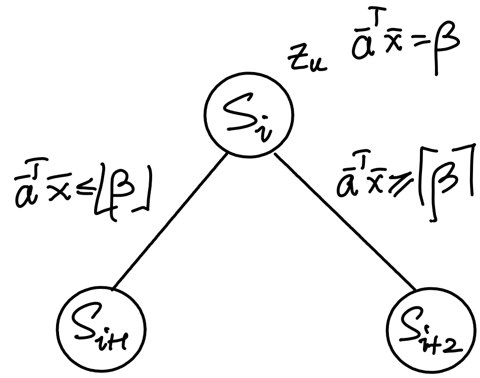
MATH 567 : Lecture 16 (03/04/2025)

Today: * branching on constraint
* Jeroslow's IP
* cutting planes

Types of Branching (continued..)

③ Binary branching on a constraint:

Assume IP, i.e., $\bar{x} \in \mathbb{Z}^n$ is required.
Let $\bar{a}^T\bar{x} = \beta$ is valid for the LP relaxation at S_i, where β is non-integral, but $\bar{a} \in \mathbb{Z}^n$. We then create two nodes by adding $\bar{a}^T\bar{x} \le \lfloor \beta \rfloor$ and $\bar{a}^T\bar{x} \ge \lceil \beta \rceil$.



<u>Example</u> Let $2x_1 + 3x_2 + 5x_3 = 7.43$ hold for LP at S_i, where $x_1, x_2, x_3 \in \mathbb{Z}$. Create two branches by adding
$2x_1 + 3x_2 + 5x_3 \le 7$ and $2x_1 + 3x_2 + 5x_3 \ge 8$.

④ Integer branching on a constraint:

Similar to ②, but for a constraint $\bar{a}^T\bar{x}$, $\bar{a} \in \mathbb{Z}^n$.

<u>Example</u> Let $6.71 \le 3x_1 + 5x_3 - x_4 + 2x_5 \le 11.99$ be valid, where $x_1, x_3, x_4, x_5 \in \mathbb{Z}$ is required. We create five branches by adding $3x_1 + 5x_3 - x_4 + 2x_5 = \beta$ for $\beta = 7, 8, 9, 10, 11$ $(7 = \lceil 6.71 \rceil,\ 11 = \lfloor 11.99 \rfloor)$.

Jeroslow's IP (1974)

$$\begin{aligned} \min \quad & x_{n+1} \\ \text{s.t.} \quad & 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n \quad \text{for } \textbf{odd } n \\ & x_j \in \{0,1\},\ j = 1,2,\ldots,n+1. \end{aligned}$$

The optimal solution must set $x_{n+1} = 1$. But binary branching on variables (option ①) will take an exponential number (in n) of nodes to solve it!

Feasibility version of Jeroslow's IP

We will prove the above result for this simpler version.

$n = 2k+1$ (odd). Consider the following feasibility binary IP:

$$\begin{aligned} 2x_1 + 2x_2 + \cdots + 2x_n &= 2k+1 \\ x_j \in \{0,1\},\ & j = 1,2,\ldots,n. \end{aligned}$$

The goal here is to **prove** that the above IP is integer infeasible using B&B.

Say, $x_1, \ldots, x_j$ for $j < k$ are fixed already (WLOG). Also, assume $x_r = 1$ for $r = 1, \ldots, i$ for $i < j$, and $x_r = 0$ for $r = i+1, \ldots, j$.

The LP feasibility problem at the current node is

$$\begin{aligned} 2x_{j+1} + \cdots + 2x_{2k+1} &= 2k+1-2i. \\ 0 \le x_r \le 1,\ & r = j+1, \ldots, 2k+1. \end{aligned}$$

$$\Rightarrow \qquad \underbrace{x_{j+1} + \cdots + x_{2k+1}}_{2k+1-j} = \frac{2k+1-2i}{2} = k - i + \frac{1}{2}$$

$$0 \le x_r \le 1, \quad r = j+1, \ldots, 2k+1.$$

As long as $j < k$, $2k+1-j > k+1$. So we can always find an LP-feasible solution (non-integral) to this subproblem. Hence we cannot prune this node! In fact, there may be many LP-feasible solutions.

$\Rightarrow$ We have to fix at least $k = \lfloor \frac{n}{2} \rfloor$ of the x_i's before we can prune a node. Hence the B&B tree has at least $2^k = 2^{\lfloor n/2 \rfloor}$ nodes.

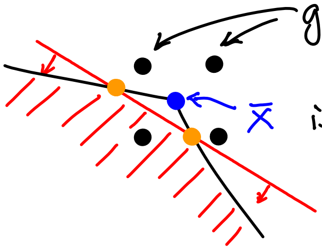
But we could prove integer infeasibility at the root node itself by branching on the constraint $x_1 + x_2 + \cdots + x_n$!

$$\begin{aligned} \max/\min \quad & \sum_{j=1}^{n} x_j \\ \text{s.t.} \quad & 2 \sum_{j=1}^{n} x_j = 2k+1 \\ & 0 \le x_j \le 1, \; j = 1, 2, \ldots, 2k+1. \end{aligned}$$

$\delta(\min) = \gamma(\max) = k + \frac{1}{2}$, and hence $\lceil \delta \rceil > \lfloor \gamma \rfloor$.

So we create zero nodes!

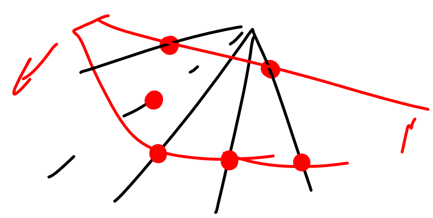
Cutting Planes



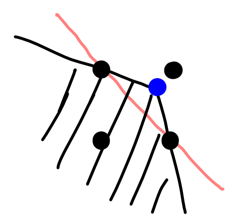
A cutting plane cuts off a non-integral corner point (in the feasible region of the LP relaxation).

As illustrated here, cutting a fractional corner point might add more (new) fractional vertices.

In higher dimensions, it could add many more new nonintegral vertices!



But it could happen that we do not add any new non-integal vertices by adding the cut. Indeed, such cuts are the tightest ones you could add.



It may not be possible to always add a tightest cut — we still benefit from cutting off fractional corner points, so we study cutting planes in general...

Recall: $\bar{a}^T\bar{x} \leq \beta$ is valid for $P \subseteq \mathbb{R}^n$ if $\bar{a}^T\bar{x} \leq \beta \;\forall\, \bar{x} \in P$.

Chvátal-Gomory (CG) Cuts

Vásek Chvátal ("Vashek HoTal").

Most other classes of cuts could be derived by applying the CG cut procedure repeatedly.

Pure integer case

(1) $Y = \{ \bar{x} \in \mathbb{Z}^n \mid A\bar{x} \le \bar{b} \}$.

$\bar{u} \geq \bar{0} \implies (\bar{u}^T A)\bar{x} \le \bar{u}^T \bar{b}$ is valid for Y.

If $\bar{u}^T A$ is integral, then

$(\bar{u}^T A)\bar{x} \le \lfloor \bar{u}^T \bar{b} \rfloor$ is valid for Y.

(2) $P = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} \le \bar{b}, \bar{x} \geq \bar{0} \}$.

$\bar{u} \geq \bar{0} \implies \underbrace{(\bar{u}^T A)\bar{x} \le \bar{u}^T \bar{b}}_{(1)}$ is valid for P.

$\implies \underbrace{\lfloor (\bar{u}^T A) \rfloor \bar{x} \le \bar{u}^T \bar{b}}_{(2)}$ is valid for P

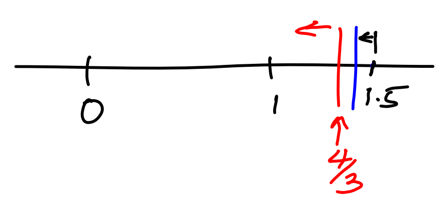
Since $\bar{x} \geq \bar{0}$, (2) weakens (1).

Hence $\lfloor \bar{u}^T A \rfloor \bar{x} \le \lfloor \bar{u}^T \bar{b} \rfloor$ is valid for $Y = P \cap \mathbb{Z}^n$.

Example $\lfloor 3.3 \rfloor x \le 4.5$ is valid for P

$\implies 3x \le \lfloor 4.5 \rfloor$ is valid for P

$3x \le 4$ is valid for $P \cap \mathbb{Z}$.

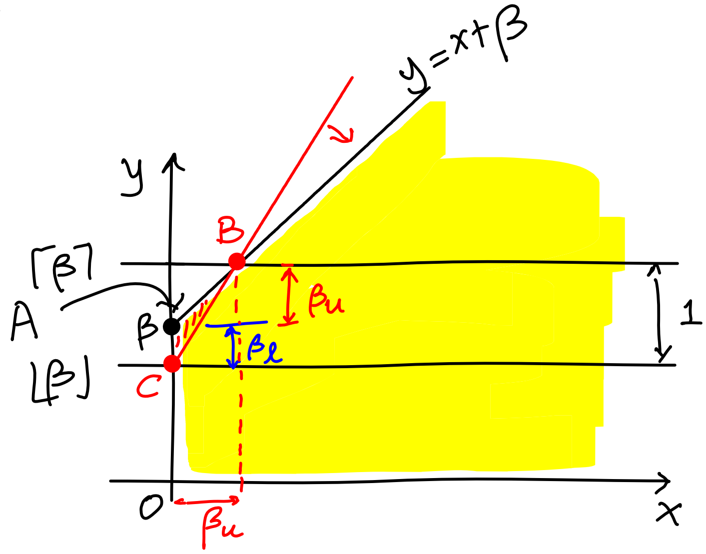


Mixed Integer Case (of the CG cut)

Mixed integer rounding (MIR)

$X = \{(x,y) \in \mathbb{R}_{\geq 0} \times \mathbb{Z} \mid y \leq x + \beta\}$.

interesting case: β is non-integral.



$X = \{(x,y) \in \mathbb{R}_{\geq 0} \times \mathbb{Z} \mid y \leq x + \beta\}$

β is non-integral.

$A(0,\beta)$ needs to be cut off.

Notation

$\beta_\ell = \beta - \lfloor \beta \rfloor$
$\beta_u = \lceil \beta \rceil - \beta$ } lower and upper fractional parts.

e.g., $\beta = 13.3$,
$\beta_\ell = 0.3$, $\beta_u = 0.7$.

Hence we get that

$$y \leq \frac{1}{\beta_u} x + \lfloor \beta \rfloor$$

is valid for X.

$x \geq 0$ is needed to get the fractional corner point $A(0,\beta)$ in the first place.

At B, $y = \lceil \beta \rceil = \beta + \beta_u$. With $y = x + \beta$, we get

$\beta + \beta_u = x + \beta \Rightarrow x = \beta_u$.

The cut is $y = mx + \lfloor \beta \rfloor$, and at $B(\beta_u, \lceil \beta \rceil)$, we get

$$\lceil \beta \rceil = \lfloor \beta \rfloor + 1 = m\beta_u + \lfloor \beta \rfloor$$

$$\Longrightarrow m = \frac{1}{\beta_u}.$$

Alternatively, $m = \frac{1}{\beta_u}$ directly from the figure