MATH 567 : Lecture 17 (03/06/2025)

Today : * MIG cuts
* Knapsack cuts
* cover inequalities

Mixed-integer Gomory Cut (MIG cut)

Extension of MIR to higher dimensions. Let

$$X = \left\{ (\bar{x}, \bar{y}) \in \mathbb{R}^m_{\geq 0} \times \mathbb{Z}^n_{\geq 0} \;\middle|\; \sum_{j \in N} a_j y_j + \sum_{j \in M} a_j x_j = \beta \right\}$$

where $N = \{1, 2, \dots, n\}$ and $M = \{1, 2, \dots, m\} + n$.

$\to$ N, M are index sets for $\bar{y}, \bar{x}$, resp.

IDEA: Derive a valid inequality of the form $y \leq x + \beta$ from the equation defining X, and apply MIR.

$$\sum_{j \in N} a_j y_j + \sum_{j \in M} a_j x_j = \beta$$

we ignore terms with $a_j \geq 0$ ($x_j \geq 0 \;\forall j$)

$$\Rightarrow \sum_{j : a_{j_\ell} \leq \beta_\ell} \lfloor a_j \rfloor y_j + \sum_{j : a_{j_\ell} > \beta_\ell} a_j y_j + \sum_{a_j < 0} a_j x_j \leq \beta$$

$\lceil a_j \rceil - a_{j_u}$

$$\Rightarrow \underbrace{\left(\sum_{j : a_{j_\ell} \leq \beta_\ell} \lfloor a_j \rfloor y_j + \sum_{j : a_{j_\ell} > \beta_\ell} \lceil a_j \rceil y_j \right)}_{y} \leq \beta + \underbrace{\left(\sum_{j : a_{j_\ell} > \beta_\ell} a_{j_u} y_j - \sum_{a_j < 0} a_j x_j \right)}_{x}$$

We now apply MIR to get $y \leq \frac{1}{\beta_u} x + \lfloor \beta \rfloor$.

$$\Rightarrow \left(\sum_{a_{j_\ell} \le \beta_\ell}' \lfloor a_j \rfloor y_j + \sum_{a_{j_\ell} > \beta_\ell}' \underbrace{\lceil a_j \rceil}_{\lfloor a_j \rfloor + 1} y_j\right) \le \lfloor \beta \rfloor + \left(\sum_{a_{j_\ell} > \beta_\ell}' \frac{a_{ju}}{\beta_u} y_j - \sum_{a_j < 0}' \frac{a_j}{\beta_u} x_j\right)$$

$$\Rightarrow \sum_{a_{j_\ell} \le \beta_\ell}' \lfloor a_j \rfloor y_j + \sum_{a_{j_\ell} > \beta_\ell}' \left(\lfloor a_j \rfloor + \left(\frac{\beta_u - a_{ju}}{\beta_u}\right)\right) y_j + \frac{1}{\beta_u} \sum_{a_j < 0}' a_j x_j \le \lfloor \beta \rfloor$$

is the MIG cut.

<u>Note</u>: If $|M| = 0$, i.e., there are no x_j's, the usual CG cut

$$\sum_{j \in N}' \lfloor a_j \rfloor y_j \le \lfloor \beta \rfloor .$$

But the MIG cut gives

$$\sum_{a_{j_\ell} \le \beta_\ell}' \lfloor a_j \rfloor y_j + \sum_{a_{j_\ell} > \beta_\ell}' \left(\lfloor a_j \rfloor + \underbrace{\left(\frac{\beta_u - a_{ju}}{\beta_u}\right)}_{\geq 0}\right) y_j \le \lfloor \beta \rfloor ,$$

which is stronger than the CG cut.

Wolsey (Integer Programming) calls the MIR cut as the "basic mixed integer inequality", and the special case of MIG cut with $|M| = 1$, $|N| = 2$ as the "MIR inequality".

Example

$$X = \left\{ (\bar{x}, \bar{y}) \in \mathbb{R}^2_{\geq 0} \times \mathbb{Z}^3_{\geq 0} \;\middle|\; \underbrace{2}_{a_4} x_1 \underbrace{-}_{a_5} x_2 + \underbrace{\tfrac{10}{3}}_{a_1} y_1 + \underbrace{}_{a_2} y_2 + \underbrace{\tfrac{11}{4}}_{a_3} y_3 = \underbrace{\tfrac{21}{2}}_{\beta} \right\}.$$

$$a_{1_\ell} = \frac{1}{3},\ a_{1_u} = \frac{2}{3},\ a_{3_\ell} = \frac{3}{4},\ a_{3_u} = \frac{1}{4},\ \beta_\ell = \beta_u = \frac{1}{2}.$$

$\Rightarrow \frac{10}{3} y_1 + y_2 + \frac{11}{4} y_4 - x_2 \leq \frac{21}{2}$ is valid for X.

Note that we have removed the $2x_1$ term from lhs.

$$\Rightarrow \left\lfloor \frac{10}{3} \right\rfloor y_1 + y_2 + \left(\left\lfloor \frac{11}{4} \right\rfloor + \frac{\left(\frac{1}{2} - \frac{1}{4}\right)}{1/2} \right) y_3 - \frac{1}{1/2} x_2 \leq \left\lfloor \frac{21}{2} \right\rfloor$$

is valid for X.

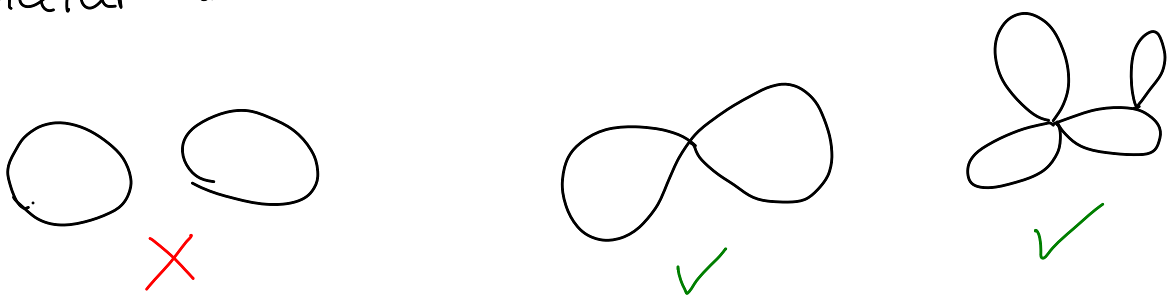
$\Rightarrow 3y_1 + y_2 + \frac{5}{2} y_3 - 2x_2 \leq 10$ is valid for X.

Project 1: Hiker's tour problem (HTP)

$G = (V, E)$ directed graph

Find a circuit (closed walk) with following properties:

* start and end at a given vertex;
* do not have to visit every $v \in V$;
* could visit a node more than once;
* subtours are allowed as long as they are connected at vertices.



* $\sum_{(i,j)\in W} c_{ij} \geq L$ ← data

Come up with formulations similar to the MTZ and subtour formulations for TSP.

Knapsack Cuts for pure 0-1 Programs

IDEA:

Given $\left\{ \begin{array}{ll} \max & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x} \le \bar{b} \\ & \bar{x} \in \{0,1\}^n \end{array} \right\}$, pick $\bar{a}^T\bar{x} \le \beta$ from $A\bar{x} \le \bar{b}$,

generate cuts for $Y = \{ \bar{x} \in \{0,1\}^n \mid \bar{a}^T\bar{x} \le \beta \}$, and add these cuts to the original IP.

Assume $a_i, \beta \in \mathbb{Z}$. WLOG, assume $a_i \ge 0 \; \forall i$ (in $\bar{a}$).

If $a_1 < 0$, we could replace x_1 with $(1-x_1)$ and a_1 with $-a_1$ to get another inequality, for instance.

We define covers that capture the subsets of a_i that add to values larger than β (and hence "cover" it). If their sum is $> \beta$, we cannot have all the corresponding $x_j = 1$, which is the cut we are seeking.

Def $C \subseteq \{1, 2, \ldots, n\} = N$ is a **cover** if $\bar{a}(C) > \beta$, where $\bar{a}(C) = \sum_{i \in C} a_i$. Further, we say that C is a **minimal cover** if C is a cover, but $C \setminus \{i\}$ is not a cover $\forall i \in C$.

Example

Let $Y = \{\bar{x} \in \{0,1\}^7 \mid \underline{11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19}\}$ (*)

$C_1 = \{1, 4, 5\}$ is a minimal cover.

$C_2 = \{3, 4, 5, 6\}$ is a minimal cover.

$C_3 = \{3, 4, 5, 6, 7\}$ is a cover, but is not minimal.

Note that $a_3 + a_4 + a_5 + a_6 + a_7 = 21 > \beta = 19$. But so is $a_3 + a_4 + a_5 + a_6 (= 20)$.

Claim C is a cover $\Longrightarrow$ $\bar{x}(C) \leq |C| - 1$ is valid for Y.

Here, $\bar{x}(C) = \sum_{j \in C} x_j$.

C_1: $x_1 + x_4 + x_5 \leq 2$ is valid for Y. — (1)

C_2: $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for Y — (2)

C_3: $x_3 + x_4 + x_5 + x_6 + x_7 \leq 4$ is valid for Y, — (3).

But (3) is weaker than (2), e.g., $x_3 + x_4 + x_5 + x_6 = 3.5$ satisfies (3), but violates (2).

Notice we added an extra variable (x_7) to the lhs of the $\leq$ inequality with $x_7 \geq 0$, but also increased the rhs by 1. If we could add more nonnegative terms to the lhs while not changing the rhs, then we will strengthen the cut.

Def The **extension** of a cover C is

$$E(C) = \{ j \notin C, j \in N \mid a_j \geq \max_{i \in C} \{a_i\} \} \cup C.$$

e.g., $E(\{3,4,5,6\}) = \{1,2,3,4,5,6\}$.

Claim $\bar{x}(E(C)) \leq |C| - 1$ is valid for Y.

So, $x_3 + x_4 + x_5 + x_6 \leq 3$ — (2) can be strengthened

to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ — (4).

Since $\bar{a}(C) > \beta$, and the added to C to obtain $E(C)$ are such that $a_j \geq \max_{i \in C}(a_i)$, the validity of the new cut follows.