MATH 567: Lecture 18 (03/18/2025)

Today: * lifted cover inequalities
* separation problem

Recall definitions on knapsack cover inequalities:

Def $C \subseteq \{1,2,\dots,n\} = N$ is a **cover** if $\bar{a}(C) > \beta$, where $\bar{a}(C) = \sum_{i \in C} a_i$. Further, we say that C is a **minimal cover** if C is a cover, but $C \setminus \{i\}$ is not a cover $\forall i \in C$.

Let $Y = \{\bar{x} \in \{0,1\}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$ (*)

$C_1 = \{1,4,5\}$ is a minimal cover.

$C_2 = \{3,4,5,6\}$ is a minimal cover.

$C_3 = \{3,4,5,6,7\}$ is a cover, but is not minimal.

Claim C is a cover $\Rightarrow$ $\bar{x}(C) \leq |C| - 1$ is valid for Y.

Here, $\bar{x}(C) = \sum_{j \in C} x_j$.

C_1: $x_1 + x_4 + x_5 \leq 2$ is valid for Y. (1)

C_2: $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for Y (2)

C_3: $x_3 + x_4 + x_5 + x_6 + x_7 \leq 4$ is valid for Y, (3)

Def The **extension** of a cover C is

$$E(C) = \{j \notin C, j \in N \mid a_j \geq \max_{i \in C} \{a_i\}\} \cup C.$$

e.g., $E(\{3,4,5,6\}) = \{1,2,3,4,5,6\}$.

Claim $\bar{x}(E(C)) \leq |C| - 1$ is valid for Y.

So, $x_3+x_4+x_5+x_6 \le 3$ —— (2) can be strengthened to

$$x_1+x_2+x_3+x_4+x_5+x_6 \le 3 \quad \text{———} (4).$$

This is an extended cover cut/inequality.

But, $\underbrace{2x_1+x_2+x_3+x_4+x_5+x_6 \le 3}_{(5)}$ is also valid for Y.

Recall, $Y = \{\bar{x} \in \{0,1\}^7 \mid 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19\}$ ⊛

(5) holds, as $x_1 = 1 \Rightarrow (x_2+\cdots+x_6) \le 1$ $(19-11=8)$.

Note that (5) is stronger than (4).

How did we get (5)? By **lifting** coefficient(s)!

We lifted the coefficient of x_1 from 1 to 2.

In a more general setting, we could lift the coefficient of some x_j from 0 to the largest possible value. Also, the idea of lifting could be applied to other classes of inequalities as well, and not just for covers.

Given a cover C_1 with $1 \notin C_1$, we know $\bar{x}(C) \le |C|-1$ is a valid inequality for $(\bar{a}^T\bar{x})(C_1) \le \beta$, where $(\bar{a}^T\bar{x})(C_1) = \sum_{j \in C_1} a_j x_j$. We want α_1 such that

$$\alpha_1 x_1 + \bar{x}(C_1) \le |C|-1 \text{ is valid for } a_1x_1 + (\bar{a}^T\bar{x})(C_1) \le \beta.$$

If $x_1 = 0$, α_1 can be any valid value ($\alpha_1 \geq 0$).

If $x_1 = 1$, $\alpha_1 + \bar{x}(C) \leq |C| - 1$ should hold for all $\bar{x} \in \{0,1\}^n$ such that $a_1 + (\bar{a}^T \bar{x})(C_1) \leq \beta$.

let
$$z = \left\{ \begin{array}{ll} \max & \bar{x}(C_1) \\ \text{s.t.} & (\bar{a}^T\bar{x})(C_1) \leq \beta - a_1 \\ & \bar{x} \in \{0,1\}^n \end{array} \right\} \text{——— (KP)}$$

Then we have $z \leq |C_1| - 1 - \alpha_1 \implies \boldsymbol{\alpha_1 \leq |C| - 1 - z}$, an upper bound on α_1. The best α_1 is $|C| - 1 - z$, but by choosing $z = z_u$, the LP-relaxation objective function value of (KP), we still get a good value for α_1.

So, we set $\alpha_1 = |C| - 1 - z_u$.

In general, we do not want to solve a subproblem as an IP — always solve only LPs as subproblems.

Example

$$Y = \{\bar{x} \in \{0,1\}^7 \,|\, 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\} \text{ ——— } (*)$$

Consider $C_2 = \{3,4,5,6\} \not\ni 1$.

(KP) here is
$$z = \left\{ \begin{array}{ll} \max & x_3 + x_4 + x_5 + x_6 \\ \text{s.t.} & 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11 = 8 \\ & x_3, x_4, x_5, x_6 \in \{0,1\} \end{array} \right\}.$$

$z = 1$ here ($x_j = 1$ for any one $j \in C_2$). $\implies \alpha_1 = |C_2| - 1 - z$
$= 4 - 1 - 1 = 2.$

Solving the LP relaxation of (KP), we get

$z_u = 1.8$ ($x_6 = 1$, and $x_5 = 0.8$ or $x_4 = 0.8$) $\Rightarrow \alpha_1 = |C_2| - 1 - z_u = 1.2$, (which is still better than 1). So, the new

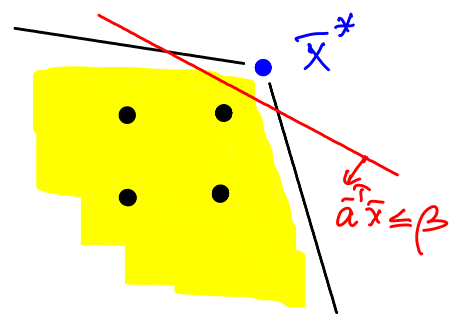
lifter cover inequality is $1.2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$.

How did I get $z_u = 1.8$? Essentially using a "greedy" approach to solve the knapsack problem.

$$\begin{aligned} \max \quad & c_1x_1 + \cdots + c_nx_n \qquad c_j, a_j \geq 0 \\ \text{s.t.} \quad & a_1x_1 + \cdots + a_nx_n \leq \beta \\ & 0 \leq x_j \leq u_j \end{aligned}$$

Sort the x_j's in the decreasing order of $\frac{c_j}{a_j}$, and set x_j's to $\min\{u_j, \beta'/a_j\}$, where β' is the "updated" β, i.e, $\beta \leftarrow \beta - a_i x_i$ after setting x_i in the previous step.

Separation Problem



In general, for any combinatorial optimization problem (COP): $\max = \{\bar{c}^T\bar{x} \mid \bar{x} \in X \subseteq \mathbb{R}^n\}$,

and given $\bar{x}^* \in \mathbb{R}^n$, is $\bar{x}^* \in \text{conv}(X)$? If YES, prove it. If NO, find an inequality $\bar{a}^T\bar{x} \leq \beta$ satisfied by all $\bar{x} \in X$, but is violated by $\bar{x}^*$, i.e, $\bar{a}^T\bar{x}^* > \beta$.

The inequality with $(\bar{a}^T\bar{x}^* - \beta)$ largest is the "most violated" separating inequality.

We consider the separation problem in the context of knapsack cover inequalities.

Let $Y = \{ \bar{x} \in \{0,1\}^n \mid \bar{a}^T \bar{x} \leq \beta \}$, $a_i, \beta \in \mathbb{Z}_{\geq 0}$, and let $\bar{x}^* \in \mathbb{R}^n$, but $\bar{x}^* \notin \{0,1\}^n$, i.e., $0 < x_j^* < 1$ for at least one $j \in N$. We want to separate $\bar{x}^*$ using a cover inequality, i.e., find a cover C such that $(\bar{a}^T \bar{x}^*)(C) > \beta$ and $\bar{x}^*(C) > |C| - 1$.

Define $\bar{y} \in \{0,1\}^n$ as the incidence vector of C. We need

$$\left\{ \begin{array}{c} \sum_{j=1}^{n} x_j^* y_j > \sum_{j=1}^{n} y_j - 1 \\ \sum_{j=1}^{n} a_j y_j > \beta \\ y_j \in \{0,1\} \ \forall j \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{c} 1 > \sum_{j=1}^{n} (1 - x_j^*) y_j \\ \sum_{j=1}^{n} a_j y_j \geq \beta + 1 \\ y_j \in \{0,1\} \ \forall j \end{array} \right\} \longrightarrow \text{as } a_j, \beta \in \mathbb{Z}_{\geq 0}$$

So, we can find

$$z = \left\{ \begin{array}{ll} \min & \sum_{j=1}^{n} (1 - x_j^*) y_j \\ \text{s.t.} & \sum_{j=1}^{n} a_j y_j \geq \beta + 1 \\ & y_j \in \{0,1\} \ \forall j \end{array} \right\}.$$

If $z<1$, the cover we seek exists, and its incidence vector is given by $\bar{y}$. Hence $\bar{x}^*$ violates the cover inequality $\bar{x}(C) \leq |C|-1$.

<u>Example</u>

$$Y = \{\bar{x} \in \{0,1\}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\} \quad (*)$$

Let $\bar{x}^* = [\overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{1}, \overset{5}{1}, \overset{6}{\tfrac{3}{4}}, \overset{7}{0}]^T$. Find a separating cover for $\bar{x}^*$.

We solve

$$\min \quad z = y_1 + y_2 + \tfrac{1}{4} y_6 + y_7$$
$$\text{s.t.} \quad 11y_1 + 6y_2 + 6y_3 + 5y_4 + 5y_5 + 4y_6 + y_7 \geq 20$$
$$y_j \in \{0,1\}, \; j = 1, \ldots, 7.$$

Optimal solution: $\bar{y} = [0,0,1,1,1,1,0]$, $z^* = \frac{1}{4}$.

Hence $\bar{x}^*$ violates $x_3 + x_4 + x_5 + x_6 \leq 3$.

Indeed, $\bar{x}^*(C) = 3\frac{3}{4} \not\leq 3$.

Note that a greedy approach gives the optimal integer solution for this knapsack problem!