MATH 567: Lecture 19 (03/20/2025)

Today: * disjunctive cuts

Disjunctive Cuts (for 0-1 IPs)

IDEA: Derive cuts by first creating a non-linear system then linearizing the same by going to higher dimensions, and then projecting back.

$$\text{If } x_j \in \{0,1\} \ \forall j, \quad x_j^2 \leftarrow x_j, \quad x_i x_j \longleftrightarrow y_{ij} \in \{0,1\}.$$

Let $P_i = \{\bar{x} \mid A_i \bar{x} \le \bar{b}^i\}$, $i = 1,2$, $P_i \ne \emptyset$.

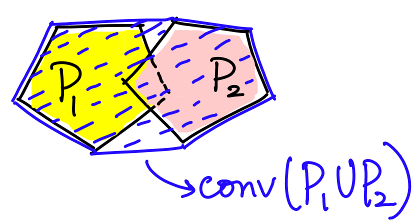
Assume $\text{rec}(P_1) = \text{rec}(P_2)$. Thus the sharp representation for $P_1 \cup P_2$ is

$$\left.\begin{aligned} A_1 \bar{x}^1 &\le \bar{b}^1 y_1 \\ A_2 \bar{x}^2 &\le \bar{b}^2 y_2 \\ \bar{x} &= \bar{x}^1 + \bar{x}^2 \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\in \{0,1\} \end{aligned}\right\} \text{— } *\text{-sharp}$$

Recall ① $\bar{x} \in P_1 \cup P_2 \iff \exists\, (\bar{x}^1, \bar{x}^2, y_1, y_2)$ such that $(\bar{x}, \bar{x}^1, \bar{x}^2, y_1, y_2)$ satisfies $*$-sharp.

② $\text{Proj}_{\bar{x}}(\text{LP relaxation of } *\text{-sharp}) = \text{conv}(P_1 \cup P_2)$.

if you project out $\bar{x}^1, \bar{x}^2, y_1, y_2$



Idea: we create a non linear system from the original system, then linearize by adding more variables, and finally project back to the original space to derive valid inequalities.

Lovász-Schrijver (LS) Procedure (for 0-1 IPs)

$$X = \{\bar{x} \in \mathbb{Z}^n \mid A\bar{x} \leq \bar{b}\}$$ includes $0 \leq x_j \leq 1$

$$K = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \leq \bar{b}\}$$

1. Select $j \in \{1, 2, \ldots, n\}$.

2. Create the non-linear system

$$\left.\begin{array}{r} (A\bar{x} - \bar{b})x_j \leq \bar{0} \\ (A\bar{x} - \bar{b})(1 - x_j) \leq \bar{0} \\ x_j(1 - x_j) = 0 \end{array}\right\} M_j^{NL}(K)$$

holds, as $x_j \in \{0, 1\}$

non linear, as there are quadratic terms $x_i x_j$

3. Linearize the system by replacing x_j^2 by x_j, and $x_i x_j$ for $j \neq i$ by y_i (where y_i is supposed to be binary). The polyhedron thus obtained is $M_j(K)$.

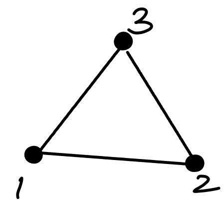
4. Let $P_j(K) = \mathrm{Proj}_{\bar{x}}(M_j(K))$. has the cut(s) we seek

The LS and other similar procedures have many theoretical and computational applications. A standard question is whether we could get the required cut by a small (i.e., polynomial) number of applications (repeatedly) of the LS procedure.

Example: Vertex packing problem (also called the maximal independent set problem — select the largest subset of vertices so that no two of the vertices are joined by an edge).

e.g.,

$$X \left\{ \begin{array}{l} \left.\begin{array}{l} x_1 + x_2 \le 1 \\ x_2 + x_3 \le 1 \\ x_1 + x_3 \le 1 \\ 0 \le x_i \le 1, \ i = 1,2,3 \end{array}\right\} K \\ \quad x_i \in \mathbb{Z} \end{array} \right.$$



Want to derive $x_1 + x_2 + x_3 \le 1$

Apply LS procedure with $j=1$:

$$x_1^2 + x_1 x_2 \le x_1 \quad , \text{ replace } x_1^2 \text{ by } x_1$$

$$\Longrightarrow \ x_1 x_2 \le 0$$

But from $-x_2 \le 0$, we get $-x_1 x_2 \le 0 \Rightarrow x_1 x_2 \ge 0.$

$$\Longrightarrow x_1 x_2 = 0.$$

Similarly, $x_1 x_3 = 0$.

Consider $(x_2 + x_3 \le 1)(1 - x_1)$:

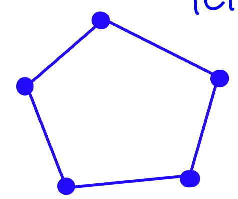
$$x_2 + x_3 - \underbrace{\cancel{x_1 x_2}}_{=0} - \underbrace{\cancel{x_1 x_3}}_{=0} \le 1 - x_1$$

$$\Longrightarrow \quad \boxed{x_1 + x_2 + x_3 \le 1}$$

This is an instance of "odd-hole" inequality.

Def An **odd hole** is $C \subseteq V$ with $|C|$ odd, with edges connecting the vertices making a simple cycle, i.e., a "hole".

$|C|=5$



We can pick at most $\frac{|C|-1}{2}$ nodes, i.e.,

$$\bar{x}(C) \leq \frac{|C|-1}{2}$$

is valid, and is derivable by the LS procedure using $M_j(C)$ for any $j \in C$.

$\sum_{(i,j)\in C}' x_{ij} \leq 2$ here.

We could also derive this inequality by adding $x_i + x_j \leq 1$ over C, which gives

$$2\,\bar{x}(C) \leq |C|,$$

which we can divided by 2, and round down (CG procedure) to get

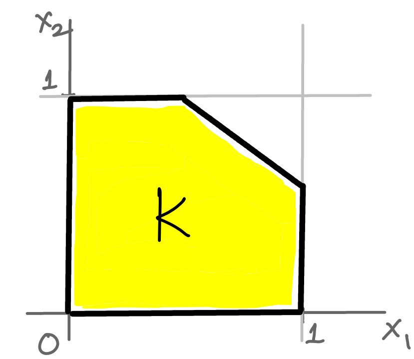
$$\bar{x}(C) \leq \left\lfloor \frac{|C|}{2} \right\rfloor = \frac{|C|-1}{2}.$$

But there are other problem instances where the LS procedure gives inequalities which cannot be derived by other procedures.

Theorem 14 If $Q_j(K) = \operatorname{conv}\left([K \cap (x_j=0)] \cup [K \cap (x_j=1)]\right)$ then $P_j(K) = Q_j(K)$.

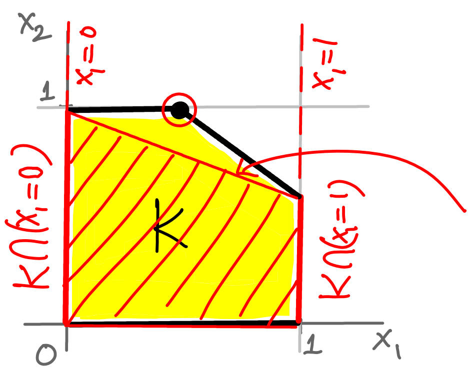
Theorem 13 was on reduced cost fixing in lecture 15!

Before presenting the proof, we illustrate the concept in 2D. Consider a non-trivial polytope in the unit square.



Consider $Q_1(K) = \operatorname{conv}\left([K \cap (x_1=0)] \cup [K \cap (x_1=1)]\right)$.

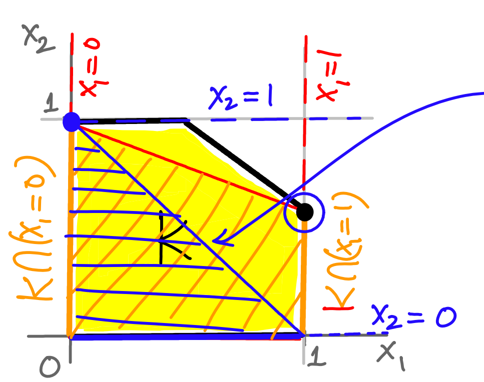
Note: the polytope need not be "symmetric" — or show the same behavior



$Q_1(K) = \operatorname{conv}\left([K \cap (x_1=0)] \cup [K \cap (x_1=1)]\right)$

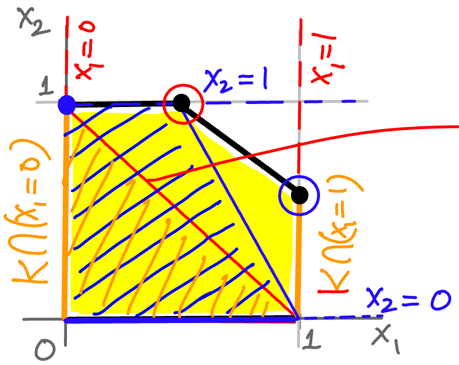
Notice how the fractional point ⊙ is cut off.

We could now apply the same procedure again using $j=2$ to get the tightest polytope. In detail, we consider $Q_2(Q_1(K))$.



$Q_2(Q_1(K))$

Notice that the other fractional corner point ⊙ is also cutoff now.



$Q_1(Q_2(K)) = Q_2(Q_1(K))$

the order does not matter!

Recall Theorem 14: $P_j(K) = \underset{j}{Q}(K) := \text{conv}\big([K\cap(x_j=0)] \cup [K\cap(x_j=1)]\big)$.

Proof ($\Leftarrow$) $\quad P_j(K) \supseteq Q_j(K)$

We try to show $K\cap(x_j=0) \subseteq P_j(K)$

and $K\cap(x_j=1) \subseteq P_j(K)$.

If $\bar{x}' \in K$ and x_j' is either 0 or 1, then

$$\left.\begin{array}{l} A\bar{x}' \le \bar{b} \\ x_j' \geq 0,\ 1-x_j' \geq 0 \\ x_j'(1-x_j') = 0 \end{array}\right\} \text{ all hold.}$$

So, we can indeed form the system $M_j^{N2}(K)$

$$\left\{\begin{array}{l} (A\bar{x}-\bar{b})x_j' \le \bar{0} \\ (A\bar{x}-\bar{b})(1-x_j') \le \bar{0} \\ x_j'(1-x_j') = 0 \end{array}\right\},$$

eliminate nonlinear terms, linearize, and project to get $P_j(K)$.

$$\Rightarrow \quad Q_j(K) \subseteq P_j(K).$$

$(\Longrightarrow)\quad P_j(K) \subseteq Q_j(K).$

We show that $P_j(K)$ contains the sharp formulation of union of polyhedra, whose convex hull is $Q_j(K)$.

$$M_j(K) \text{ has } \left\{ \begin{array}{c} (A\bar{x}-\bar{b})x_j \le \bar{0} \\ (A\bar{x}-\bar{b})(1-x_j) \le \bar{0} \\ x_j(1-x_j) = 0 \end{array} \right\} \qquad \begin{array}{c} A\overbrace{\bar{x}x_j}^{\bar{x}^1} - \bar{b}\overset{y_1}{x_j} \le \bar{0} \\ A\overbrace{\bar{x}(1-x_j)}^{\bar{x}^2} - b\underbrace{(1-x_j)}_{}^{y_2} \le \bar{0} \\ [\bar{x}(1-x_j)]_j = 0 \end{array}$$

Write $\bar{x}x_j$ as $\bar{x}^1$, $\bar{x}(1-x_j)$ as $\bar{x}^2$, $x_j \leftarrow y_1$, $(1-x_j) \leftarrow y_2$

$$\Longrightarrow \quad A\bar{x}^1 \le \bar{b}y_1$$
$$A\bar{x}^2 \le \bar{b}y_2$$

$$\bar{x} - \bar{x}x_j - \bar{x}(1-x_j) = \bar{0} \Rightarrow \bar{x} = \bar{x}^1 + \bar{x}^2$$

$$x_j + (1-x_j) = 1 \Longrightarrow y_1 + y_2 = 1$$

$$x_j^2 - x_j = 0 \Rightarrow (\bar{x}^1)_j = y_1 \equiv \overset{j^{th}\text{ unit vector}}{\bar{e}_j^T}\bar{x}^1 = y_1$$

$$[\bar{x}(1-x_j)]_j = 0 \Rightarrow \bar{x}_j^2 = 0 \equiv \bar{e}_j^T\bar{x}^2 = 0 = 0\cdot y_2$$

$$\boxed{\begin{array}{l} A\bar{x}^1 \le \bar{b}y_1 \\ \bar{e}_j^T\bar{x}^1 = y_1 \\ A\bar{x}^2 \le \bar{b}y_2 \\ \bar{e}_j^T\bar{x}^2 = 0\cdot y_2 \\ \bar{x} = \bar{x}^1 + \bar{x}^2 \\ y_1 + y_2 = 1 \end{array}}$$

polyhedron of the sharp formulation of $P_1 \cup P_2$

where $P_1 = \{\bar{x} \mid A\bar{x} \le \bar{b},\ \underbrace{\bar{e}_j^T\bar{x} = 1}_{x_j = 1}\}$ and $P_2 = \{\bar{x} \mid A\bar{x} \le \bar{b},\ \underbrace{\bar{e}_j^T\bar{x} = 0}_{x_j = 0}\}$.

$$\Longrightarrow \quad P_j(K) \subseteq Q_j(K).$$

$\square$